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APRIL 1968

INTRODUCTION TO THREE-DIMENSIONAL BOUNDARY LAYERS

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PREFACE

This Memorandum is the result of a study carried out at RAND during the summer of 1965. The objective was two-fold: to assess the efficacy of the theory existing at that time, and to brief RAND personnel on a branch of fluid mechanics which is ordinarily overlooked in texts and graduate curricula. A short series of lectures accompanied the development of this Memorandum. The response to this indicated that publication of this Memorandum would be worthwhile as part of a continuing RAND program, under U.S. Air Force Project RAND, of surveying the state of the art in aerodynamics.

A framework for the description of three-dimensional flow fields is a requisite for the rational analysis and design of aerodynamic systems. This Memorandum focuses on one topic in this general area, three-dimensional boundary layer phenomena, and introduces the concepts and techniques of this subject to those workers in aerodynamics and fluid mechanics who are not yet actively engaged in its study.

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SUMMARY

The basic concept of a three-dimensional boundary layer is reviewed from both physical and mathematical viewpoints. Particular emphasis is placed on the various causes of secondary flow, with geodesic curvature of the surface streamlines of inviscid flow receiving the most detailed consideration. Various exact solutions for steady, incompressible laminar flow are reviewed and a proposal for a finite-difference scheme for arbitrary inviscid flows and surface conditions is sketched (but not developed and tested). The momentum-integral method and other approximation schemes are briefly discussed.

Compressibility effects are discussed qualitatively, and then attention is turned to the stability of laminar flows and the transition to turbulence.

The equations for time-averaged turbulent flow are derived and criticized, both in differential and integral form. We review the two-dimensional case as a reminder of the essential difficulty of indeterminacy, and then critically examine existing empirically-based models of the three-dimensional turbulent boundary layer. No attempt is made to review existing methods of computing the development of turbulent layers.

We conclude with suggestions for further study, most of which seem as interesting today as they did in 1965.

ACKNOWLEDGMENTS

My thanks are extended to Professor D. E. Coles of the California Institute of Technology, to Dr. J. Aroesty of The RAND Corporation, to Professor A. F. Charwat of the University of California at Los Angeles, my colleague Professor S. A. Berger of the University of California at Berkeley, and to Dr. M. Sherman of The RAND Corporation, for particularly active discussions or criticisms.

CONTENTS

SYMBOLS

All symbols are defined where they appear in the text, and many appear in only one section of the Memorandum. Some symbols are used with more than one meaning when their appearances are well separated in space and context, and confusion seems unlikely.

The symbols which recur throughout the text are listed below.

Independent Variables and Given Parameters

ξ, η = coordinates of an orthogonal mesh covering the body. Defined in dimensions of length

ζ = distance normal to the body

h_1, h_2 = metric coefficients for ξ and η (dimensionless)

κ_1 = geodesic curvature of the coordinate curve, η = constant, on the body (length^{-1})

κ_2 = geodesic curvature of ξ = constant curve (length^{-1})

$\vec{\omega}$ = angular velocity of reference frame relative to inertial space; ω_1, ω_2, ω_3 are its ξ-, η-, and ζ-components

R = distance from the point (ξ,η,ζ) to the axis of reference frame rotation

L = a characteristic body length

U, V = ξ- and η-components of velocity at the wall in inviscid flow

p = pressure, predetermined by the inviscid flow

Λ = potential of a conservative body force

ρ = density, assumed constant except in Section XI

P = abbreviation for $p + \rho(\Lambda - \omega^2 R^2/2)$

μ = viscosity, assumed constant except in Section XI

ν = kinematic viscosity, assumed constant except in Section XI

x,y,z = dimensionless, and in the case of z, stretched, versions of ξ, η, ζ, defined on page 13

m,n,r,s = dimensionless pressure-gradient parameters defined on pages 13 and 14

κ, λ = dimensionless coordinate-curvature parameters (proportional to κ_2, κ_1 respectively), defined on page 14

Dependent Variables

u,v,w = ξ-, η-, ζ-components of velocity in the boundary layer

f',g' = dimensionless velocity components $f' = u/U$, $g' = v/V$

G^1 = dimensionless v-velocity in intrinsic coordinate system, $G' = u/U$

$\vec{\Omega}$ = vorticity vector, with ξ-, η-, and ζ-components Ω_1, Ω_2, Ω_3

$\vec{\tau}_w$ = surface shear stress (skin friction vector), with ξ- and η-components τ_1, τ_2

I. INTRODUCTION

When one considers the high degree of three-dimensionality characteristic of almost all bodies of aerodynamic interest, it seems remarkable that the study of three-dimensional boundary layers has gained so little popularity among aerodynamicists. Physico-chemical or electromagnetic complications are frequently introduced into studies of two-dimensional or axisymmetric boundary layers, but geometrical complications are regarded with great trepidation--if regarded at all.

The possibility that this attitude not only defied practical reality but also hampered our efforts to understand fluid motions was initially brought to my attention by a lecture given by Dr. E. A. Eichelbrenner of O.N.E.R.A., by Professor M. J. Lighthill's stimulating chapter in Rosenhead's <u>Laminar Boundary Layers</u>, and by a great number of articles in the meteorological literature, especially those describing attempts to integrate the "primitive equation" as a model of large-scale atmospheric circulations.

The opportunity to investigate the matter further was provided by RAND, at the suggestion of Dr. Jerome Aroesty. My early impressions, gained from reading the survey articles listed in the bibliography, have confirmed my feeling that this topic is worthy of a more broadly distributed interest among fluid dynamicists.

The objective of this Memorandum is to provide a nonspecialist's view of the work of specialists. It is assumed that the reader is substantially familiar with two-dimensional boundary layer theory, and thus has a feeling for how the <u>magnitude</u> of the velocity vector is reduced through the boundary layer under various sorts of external flows. Our discussions of three-dimensional boundary layers will focus on how the <u>direction</u> of the velocity vector varies through the boundary layer, and hence little attention will be paid to those flows (i.e., axisymmetric, nonspinning) in which no such variation appears.

It is in this turning of the velocity vector that we discover the phenomenological richness of three-dimensional boundary-layer flows. Freed from the constraint of two-dimensional motion, the fluid seems to "come alive," seeking more or less tangential detours around

"obstacles" imposed by the pressure distribution, bringing about a fascinating topology of skin friction lines or "surface streamlines," complicating the concept of separation, and so on. We would hope to achieve from our study some feeling for how and why these things occur, perhaps some rules of thumb concerning the effects of body spin or of inviscid streamline curvature to add to those for the effects of streamwise pressure gradient.

It should be understood at the outset that this Memorandum does not constitute a review article in the usual sense. In the first place I am not personally expert in the subject treated, and have derived my information entirely from the writing of others. Topics of real practical importance, such as the effects of compressibility and of spinning bodies, have frankly not been accorded their deserved share of attention, partly by intent (to focus attention more sharply on purely geometrical complexities) and partly by the limitations of time and my acquaintance with the literature.

This Memorandum begins with a largely intuitive review (and extension to three-dimensional flow) of the concept of a boundary layer and of a properly posed problem of steady flow in boundary-layer theory. The laminar boundary-layer equations are then presented in orthogonal curvilinear coordinates. For steady, incompressible flow we introduce the Falkner transformation to eliminate the normal velocity and to state the tangential momentum equations in a form convenient for all our future discussions of laminar flow. These discussions begin with a suggestion for finite-difference solution procedures in various coordinate systems, following the method applied to two-dimensional flows by Smith and Clutter (1963), but making frequent reference to the method developed by Raetz (1957) for three-dimensional flows. Efforts to depart from the same starting point by series expansion in the surface coordinates are noted. A brief review of exact similar solutions at stagnation points and for flow over developable surfaces follows.

Next treated are the two principles that have been exploited in most solutions found to date, namely the <u>independence principle</u> for swept flows and the <u>prevalence principle</u> for cases with weak secondary flow.

Returning to a more general viewpoint, we give a semi-intuitive discussion of the causes and nature of secondary flows, and a fairly extended discussion of the topology of skin friction lines, or "surface streamlines," concluding with a definition and discussion of boundary-layer separation in three dimensions.

Momentum-integral approximation methods are discussed next, both because they lead to some useful results with a considerable savings in computational effort, and because the difficulties into which they occasionally lead us provide useful tests of our intuitive understanding of the boundary layer. The momentum-integral equations are derived directly from the Falkner-transformed equations for laminar flow, and are later re-derived more conventionally for turbulent flow, to provide a check. Velocity profile assumptions and the use of weighted integral equations are discussed at some length.

Compressibility effects are mentioned very briefly and mostly by reference in Section XI, as are laminar-flow stability in Section XII and transition to turbulence in Section XIII. The displacement thickness of a three-dimensional boundary layer is defined in Section XI.

An extended discussion of the incompressible turbulent boundary layer follows (Section XIV), with the assumption that the reader can benefit from a brief review of the problem of formulating a momentum-integral approach for two-dimensional mean flows. It is concluded that very successful three-parameter models of mean velocity profiles exist, but that methods for predicting the variation of the profile parameters are still essentially deficient. No three-parameter profile model as generally successful as the "wall-wake" model of two-dimensional flow has yet been found for the three-dimensional case.

Finally, the Memorandum concludes with a number of suggestions for experiments, calculations, or analyses which appear potentially fruitful for developing our powers of description and prediction in the areas covered.

The bibliograpy is not complete, but does include all available recent survey articles. These in turn contain comprehensive bibliographies, from which almost all the individual references given here are drawn. One gets the impression from current issues of abstracting

journals that the subject currently receives little attention in comparison with more recently glamorous subjects, so that even a 1962 or 1963 bibliography or survey article may be considered reasonably up to date. The sources which have most influenced the spirit of this Memorandum are the survey articles by Mager (1964), Head (1961), and Rotta (1962), the chapters by Lighthill (1963), the report by Raetz (1957), the paper of Smith and Clutter (1963), the N.L.L. reports of Timman (1951) and Zaat (1956), the paper of Coles (1956) and the O.N.E.R.A. publications and other papers by Eichelbrenner and co-workers.

II. THE BOUNDARY-LAYER APPROXIMATION IN THREE-DIMENSIONAL FLOW

Survey articles (e.g., Moore (1956) or Mager (1963)) usually distinguish between boundary-layer flows and boundary-region flows. In the examples usually cited, the distinction seems to be the following:

(A). In boundary-layer flows we may treat diffusion as proceeding in only one direction, normal to the wall.

This generalizes the concepts of two-dimensional boundary-layer theory, in which streamwise diffusion is neglected, to state that diffusion in all directions parallel to the wall is to be neglected.

The physical idea behind this approximation is the thinness of the boundary layer, as measured along a normal to the wall, compared to typical streamwise or lateral* distances of interest.

In boundary-region flows, exemplified by flow along a corner or edge where two walls join abruptly, there is clearly no unique normal direction at a boundary-layer point close to the corner, and hence lateral diffusion must be reinstated. These flows are not treated here.

(B). In boundary-layer flows we may take the pressure to be predetermined (on first approximation) by the inviscid-flow pattern on the surface of the given body.

The accuracy of this assumption depends upon (1) the boundary layer being thin relative to any local radius of curvature of the body, and (2) the "displacement effect" of the boundary layer producing only small perturbations in the local external flow.

Both assumptions A and B may be violated by the phenomenon of separation, if this leads to detachment of a thin shear layer the normal to which does not nearly coincide with the local normal to the wall. In such a case assumption A would misdirect the dominant diffusive flux, and the large displacement effect would violate assumption B.

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*We shall often need a word to indicate the direction, parallel to the wall, but normal to the local velocity vector. For this we choose "lateral."

III. BOUNDARY VALUES AND INITIAL DATA

The specification of boundary values for three-dimensional boundary-layer problems exactly parallels that for two-dimensional flows. We ordinarily specify no slip at the wall for the tangential components of velocity, and may prescribe an arbitrary distribution of normal velocity (suction or blowing) on the wall, so long as we do not thereby excessively thicken the boundary layer. (Roughly speaking, these normal velocities should only be of the order of magnitude of those arising spontaneously in the boundary layer in the impermeable wall case.) Temperature or heat transfer, and concentration or mass transfer must also be given at the wall.

At "infinity" with respect to a suitably stretched normal distance variable, the tangential velocity components, temperature and composition are equated to the values of these quantities appearing at the wall in the inviscid flow solution. The normal velocity is <u>not</u> specified there, but is derived from the boundary-layer solution.

As we shall discuss in more detail in Section VIII, the boundary layer in steady flow over a smoothly curved body can be said to be born at a finite number of isolated nodal points of attachment, some of which coincide with the attachment points, or "forward stagnation points," of the inviscid flow. In the immediate vicinity of such attachment points can be found a local similar solution of the boundary-layer equations, which is also an exact solution of the full Navier-Stokes equations in an infinitesimal neighborhood of the normal at the stagnation point. These special solutions can be used to establish initial data for a boundary-layer calculation in an extended region surrounding the attachment point by assigning values to all unknowns on a "wall" of normals arising out of a conveniently chosen surface curve enclosing the attachment point.

If the inviscid flow attachment is along an infinitely sharp leading edge (or an edge which is conveniently idealized as such) then the boundary layer is born all along this edge and its initial development is given by a similar solution of a Falkner-Skan type. This can be used to construct initial data on a normal wall closely paralleling

the leading edge. This procedure would also apply to the degenerate case of essentially two-dimensional flow in which the leading edge is a cylinder of finite curvature aligned exactly normal to the oncoming flow. (Think, for example, of a ring wing or axisymmetric engine intake at zero incidence.)

So far, the discussion resembles the corresponding one of two-dimensional boundary layers. Complications enter, however, when we consider the effects of three-dimensionality on the domain of dependence downstream of a finite segment of our "initial data curve."

Having adopted the assumption that, with boundary conditions given as above, the boundary layer evolves under the influence of a predetermined pressure field in a region in which diffusion takes place only along the normal to the wall, we expect to be able to construct solutions by starting with the given values on the "initial data surface" and "marching" steadily downstream, accounting at each step for the modification of the velocities, etc., by convection along the local velocity vector and diffusion along the local wall normal. An immediately practical question is the following: "Over what area of the body surface will the solution be completely determined by the given pressure distribution and inviscid velocity field, and by the solution already known over a given finite segment of the 'initial data curve'?"

In Fig. 1, $\widehat{AB}$ is the finite segment in question. We assume that $\widehat{AB}$, and the entire curve of which it is part, is oriented so that all boundary-layer particles cross it from one side to the other. This condition, which physically requires that all flow entering the region downstream of the "initial data surface" comes from a single "boundary-layer birthplace" or attachment region, amounts mathematically to the requirement that the initial data surface not be tangent to, or crossed more than once by, a characteristic curve of the boundary-layer equations (see Raetz (1957) for mathematical discussions).*

*Raetz states an "influence principle" as follows. "The influence of the solution at any point is transferred to other points first by conduction along the straight line paralleling the ζ axis [the normal] and passing through that point and then by convection downstream along all streamlines through that line." He calls the dashed lines through A and B the "outer and inner characteristic envelopes."

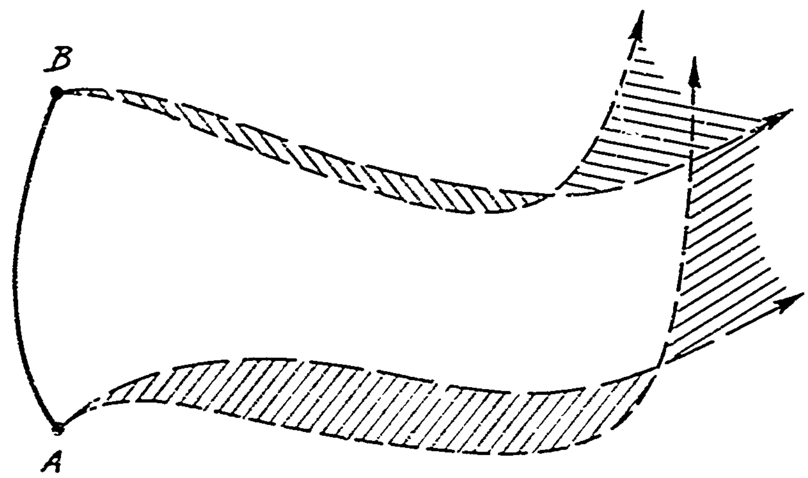

Fig. 1 -- Regions of influence and domain of dependence.

Particles which follow varied downstream trajectories will pass through any of the normals generating the surface over $\widehat{AB}$, according to their initial distance from the wall.

If we imagine the normals through A and B to be line sources of dye, the plan view of the distorting dye sheets in the downstream region would exhibit the shaded banners of the sketch.

According to our assumption of purely normal diffusion, every particle in the boundary layer over these shaded banners can become stained by this dye, whether or not it passed through the normal over A or B, but no other particles are contaminated. The shaded regions can then be called the regions of influence of the initial data lines through A and B.

Correspondingly, the boundary layer over the unshaded region between the two banners knows only about the past history of particles which have passed the initial data surface over the arc $\widehat{AB}$, and hence the unshaded area is the area referred to in the underlined question above. We shall call it the domain of dependence of the solution over the arc $\widehat{AB}$, although in the strict mathematical sense that term should include also the region upstream of $\widehat{AB}$, defined by stream surfaces which eventually converge at the attachment normal.

It may appear startling to assert that once we have adopted the boundary-layer approximations we have, in effect, no further need of the fluid which passes outboard of the lines over A and B, in order to determine the flow in our unshaded region! However, there are but two ways in which this fluid can make its presence felt. The first is by pressure, but that should be predetermined. The second is by lateral diffusion, but we have ignored that for the moment.

IV. BOUNDARY-LAYER EQUATIONS IN ORTHOGONAL COORDINATES: LAMINAR FLOW

CONTINUITY AND MOMENTUM EQUATIONS

Following the notation of Mager (1963), to whom the reader is referred for detailed discussion and references, we introduce generalized orthogonal coordinates ξ, η, ζ. The lines of constant ξ and η form a network on the body surface, and ζ increases away from the body along the normal.

The increment of distance between coordinate lines ξ and $\xi + d\xi$ is $h_1\, d\xi$; corresponding increments are $h_2\, d\eta$ and $h_3\, d\zeta$. For thin boundary layers we can assume that the metric coefficients satisfy

$$h_1 = h_1(\xi,\eta), \qquad h_2 = h_2(\xi,\eta), \qquad h_3 = 1$$

so that ζ can be interpreted as actual normal distance from the wall, which lies at $\zeta = 0$.*

The velocity components in the ξ, η, and ζ directions are called u, v, and w. The corresponding surface velocities of the inviscid flow are

$$U(\xi,\eta), \qquad V(\xi,\eta), \qquad \text{and} \qquad 0$$

The boundary-layer assumptions reduce the complex expressions for viscous stress components in a general orthogonal coordinate system to the two simple results

$$\tau_{31} = \mu \frac{\partial u}{\partial \zeta}, \qquad \tau_{32} = \mu \frac{\partial v}{\partial \zeta}$$

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*We do not, of course, have to define ζ as the normal distance itself. Raetz, for example, takes

$$\zeta = (1 - u/U)^{1/2}$$

after demanding that ξ and η be so oriented as to render u/U monotonic in normal distance.

The continuity equation is simply

$$\frac{\partial \rho}{\partial t} + \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial \xi}(h_2 u\rho) + \frac{\partial}{\partial \eta}(h_1 v\rho) + \frac{\partial}{\partial \zeta}(h_1 h_2 w\rho)\right\} = 0$$

The ξ-component of the momentum equation is

$$\frac{\partial u}{\partial t} + \frac{u}{h_1}\frac{\partial u}{\partial \xi} + \frac{v}{h_2}\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + \frac{uv}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{v^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{1}{h_1}\frac{\partial \Lambda}{\partial \xi} - 2\omega_3 v - \frac{\omega^2}{2h_1}\frac{\partial}{\partial \xi}R^2 + \frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} = \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right)$$

The η-component reads

$$\frac{\partial v}{\partial t} + \frac{u}{h_1}\frac{\partial v}{\partial \xi} + \frac{v}{h_2}\frac{\partial v}{\partial \eta} + w\frac{\partial v}{\partial \zeta} - \frac{u^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{uv}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{1}{h_2}\frac{\partial \Lambda}{\partial \eta} + 2\omega_3 u - \frac{\omega^2}{2h_2}\frac{\partial}{\partial \eta}R^2 + \frac{1}{\rho h_2}\frac{\partial p}{\partial \eta} = \frac{1}{\rho}\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right)$$

The ζ-component reads

$$\frac{\partial \Lambda}{\partial \zeta} - 2(\omega_2 u - \omega_1 v) - \frac{\omega^2}{2}\frac{\partial}{\partial \zeta}R^2 + \frac{1}{\rho}\frac{\partial p}{\partial \zeta} = 0$$

In the ξ- and η-component equation, the top lines simply give the components of acceleration, while the terms in the bottom lines represent (1) a body force derivable from the potential Λ, (2) a Coriolis force arising if the body (and hence the coordinate system, is spinning uniformly and steadily with respect to an inertial frame, (3) a centrifugal force derived from such spinning, (4) the predetermined pressure force, and (5) the viscous force.

The local ξ, η, and ζ components of the angular velocity of the body are ω_1, ω_2, and ω_3, with $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$. The distance from the spin axis to the point (ξ,η,ζ) is R.

The ζ-component equation asserts that the pressure gradient normal to the wall counterbalances the body force and the "forces" due to spinning. These forces are assumed to be sufficiently small so that throughout the thin boundary layer we may take

$$p = p(\xi,\eta)$$

with p related to the velocities U and V by

$$\frac{\partial U}{\partial t} + \frac{U}{h_1}\frac{\partial U}{\partial \xi} + \frac{V}{h_2}\frac{\partial V}{\partial \eta} + \frac{UV}{h_1 h_2}\frac{\partial h_1}{\partial \eta} - \frac{V^2}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{1}{h_1}\frac{\partial \Lambda}{\partial \xi} - 2\omega_3 V - \frac{\omega^2}{2h_1}\frac{\partial}{\partial \xi} R^2 + \frac{1}{\rho_e h_1}\frac{\partial p}{\partial \xi} = 0$$

and

$$\frac{\partial V}{\partial t} + \frac{U\partial V}{h_1\partial \xi} + \frac{V\partial V}{h_2\partial \eta} - \frac{U^2}{h_1 h_2}\frac{\partial h_1}{\partial \eta} + \frac{UV}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

$$+ \frac{1}{h_2}\frac{\partial \Lambda}{\partial \eta} + 2\omega_3 U - \frac{\omega^2}{2h_2}\frac{\partial}{\partial \eta} R^2 + \frac{1}{\rho_e h_2}\frac{\partial p}{\partial \eta} = 0$$

Here, ρ_e designates the surface density of the inviscid flow.

In compressible flows, in which ρ and μ must be treated as variable, we need also the energy equation, an equation of state, and a description of the dependence of μ and of the thermal conductivity upon the thermodynamic variables. In what follows immediately, we shall assume $\rho = \rho_e$ = constant, and μ = constant.

This assumption focuses attention on effects due to the three-dimensionality of geometrical constraints, or to Coriolis force. When

$\rho = \rho_e$, gravity and centrifugal forces act equally on boundary-layer and external flow, and hence this assumption rules out qualitatively interesting phenomena such as free convection. We shall comment briefly on these and other, primarily quantitative, effects of variable density and viscosity in Section XI.

THE FALKNER TRANSFORMATION

For discussions of exact self-similar solutions and for the formulation of systematic methods of calculation of more general boundary-layer flows, it is helpful to transform variables in some manner which accounts roughly for the anticipated magnitudes of boundary-layer thickness and velocity components. Many such transformations are available. We choose, rather arbitrarily, that associated with the name of Falkner.*

We call

$$x \equiv \frac{\xi}{L}, \qquad y \equiv \frac{\eta}{L}, \qquad z \equiv \zeta\sqrt{\frac{U(\xi,\eta)}{\nu\xi}}$$

where L is a characteristic body dimension and ν is the kinematic viscosity.** Thus

$$\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta} = \frac{1}{L}, \qquad \frac{\partial x}{\partial \eta} = \frac{\partial x}{\partial \zeta} = \frac{\partial y}{\partial \xi} = \frac{\partial y}{\partial \zeta} = 0$$

$$\frac{\partial z}{\partial \xi} = \left(\frac{m-1}{2}\right)\frac{z}{xL}, \qquad \frac{\partial z}{\partial \eta} = \left(\frac{n}{2}\right)\frac{z}{yL}, \qquad \frac{\partial z}{\partial \zeta} = \sqrt{\frac{U}{\nu L x}}$$

where

$$m(x,y) \equiv \frac{\partial \ln U}{\partial \ln x}, \qquad n(x,y) \equiv \frac{\partial \ln U}{\partial \ln y}$$

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*Laminar Boundary Layers, L. Rosenhead, ed., Oxford, 1963, p. 266 (hereafter cited in the text as L.B.L.).

**The choice of U rather than V and of ξ rather than η in the normalization of ζ implies only that ξ increases in a more-or-less downstream direction, and that $U \neq 0$ except at singular points or lines.

Later we shall also want the abbreviations

$$r(x,y) \equiv \frac{\partial \ln V}{\partial \ln x}, \qquad s(x,y) \equiv \frac{\partial \ln V}{\partial \ln y}$$

Our dependent variables are written as

$$u \equiv U(x,y) \frac{\partial}{\partial z} f(x,y,z)$$

which we designate simply $u \equiv Uf'$. At this point the prime of f does <u>not</u> imply that f depends only upon z, but is simply a shorthand. We shall write out $\partial f/\partial x$ and $\partial f/\partial y$ fully. Similarly, we write

$$v \equiv V(x,y) \frac{\partial}{\partial z} g(x,y,z) = Vg'$$

Using the continuity equation, with ρ = constant, we find, for the simple case of an impervious wall,

$$w = -\frac{1}{h_1 h_2}\sqrt{\frac{\nu x}{UL}}\left\{\frac{U}{h_1 x}\left[\left\{\left(\frac{m+1}{2}\right)+\kappa\right\} f + \left(\frac{m-1}{2}\right) z f' + x\frac{\partial f}{\partial x}\right]\right.$$

$$\left. + \frac{V}{h_2 y}\left[\left(s - \frac{n}{2} + \lambda\right) g + \frac{n}{2} z g' + y \frac{\partial g}{\partial y}\right]\right\}$$

where we have introduced our final shorthand notation

$$\kappa \equiv \frac{\partial \ln h_2}{\partial \ln x}, \qquad \lambda \equiv \frac{\partial \ln h_1}{\partial \ln y}$$

If u, v, and w are to vanish at $z = 0$, we see that boundary conditions on f and g are

$$f(x,y,0) = f'(x,y,0) = g(x,y,0) = g'(x,y,0) = 0$$

while

$$f'(x,y,\infty) = g'(x,y,\infty) = 1$$

The transformed component momentum equations, in which the pressure gradients have been eliminated in favor of inviscid-flow velocity gradients, etc., become

$$f''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \varkappa\right) ff'' + \frac{m}{h_1}(1 - f'^2) + \frac{Vx}{Uy}\frac{1}{h_2}\left\{\left(s - \frac{n}{2} + \lambda\right) gf''\right.$$

$$\left. + (n+\lambda)(1 - g'f')\right\} - \left(\frac{V}{U}\right)^2 \frac{\varkappa}{h_1}(1 - g'^2) - 2\left(\frac{\omega_3 LV}{U^2}\right) x(1 - g')$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) + \frac{V}{U}\left(\frac{1}{h_2}\right)\left(g'\frac{\partial f'}{\partial y} + f''\frac{\partial g}{\partial y}\right)\right\} \qquad (1)^*$$

and

$$g''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \varkappa\right) fg'' + \frac{1}{h_1}(r + \varkappa)(1 - f'g') + \frac{Vx}{Uy}\frac{1}{h_2}\left\{\left(s - \frac{n}{2} + \lambda\right) gg''\right.$$

$$\left. + s(1 - g'^2)\right\} - \frac{Ux}{Vy}\frac{\lambda}{h_2}(1 - f'^2) + 2\left(\frac{\omega_3 L}{V}\right) x(1 - f')$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial g'}{\partial x} - g''\frac{\partial f}{\partial x}\right) + \frac{V}{U}\left(\frac{1}{h_2}\right)\left(g'\frac{\partial g'}{\partial y} - g''\frac{\partial g}{\partial y}\right)\right\} \qquad (2)$$

THE TRANSFORMED EQUATIONS IN "INTRINSIC COORDINATES"

Of the various possible special orientations of the ζ and η axes, one of the most interesting is that in which ξ measures distance downstream along the surface streamlines of the inviscid flow. The corre-

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*For convenience, all numbered equations appearing in the text are collected in the Appendix, p. 89.

sponding surface coordinates are called "intrinsic coordinates." This choice of coordinate system implies that $V = 0$, and calls for a renormalization of v, as, for example,

$$v = U(x,y)G'(x,y,z)$$

Then $G'(x,y,0) = G'(x,y,\infty) = 0$.

The transformed momentum equations become slightly simpler, namely

$$\begin{aligned} f''' &+ \frac{1}{h_1}\left(\frac{m+1}{2} + \varkappa\right) ff'' + \frac{m}{h_1}(1 - f'^2) + \frac{x}{y}\left(\frac{1}{h_2}\right)\left\{\left(\frac{n}{2} + \lambda\right)gf''\right. \\ &\left. - (n + \lambda)G'f'\right\} + \frac{\varkappa}{h_1} G'^2 + 2\left(\frac{\omega_3 L}{U}\right)xG' \\ &= x\left\{\frac{1}{h_1}\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) + \frac{1}{h_2}\left(G'\frac{\partial f'}{\partial y} - f''\frac{\partial G}{\partial x}\right)\right\} \end{aligned} \tag{3}$$

and

$$\begin{aligned} G''' &+ \frac{1}{h_1}\left(\frac{m+1}{2} + \varkappa\right) fG'' - \left(\frac{m+\varkappa}{h_1}\right) f'G' + \frac{x}{y}\left(\frac{1}{h_2}\right)\left\{\left(\frac{n}{2} + \lambda\right) GG''\right. \\ &\left. - nG'^2 - \lambda(1 - f'^2)\right\} + 2\left(\frac{\omega_3 L}{U}\right) x(1 - f') \\ &= x\left\{\frac{1}{h_1}\left(f'\frac{\partial G'}{\partial x} - G''\frac{\partial f}{\partial x}\right) + \frac{1}{h_2}\left(G'\frac{\partial G'}{\partial y} - G''\frac{\partial G}{\partial y}\right)\right\} \end{aligned} \tag{4}$$

V. EXACT SOLUTIONS, INDEPENDENCE PRINCIPLE, AND CONDITIONS FOR ABSENCE OF SECONDARY FLOW

STAGNATION POINT SOLUTIONS

Suppose we deal with a "rounded" body, to which the flow from upstream "attaches" at some point P, which we take as the origin of coordinates. In the immediate vicinity of P we can expand the surface metric coefficients in power series in x and y, e.g.,

$$h_1(x,y) = h_1(0,0) + \left(\frac{\partial h_1}{\partial x}\right)(0,0)x + \left(\frac{\partial h_1}{\partial y}\right)(0,0)y + \ldots$$

In particular, we choose a system of coordinates that is locally rectangular at the origin, so that

$$h_1(0,0) = h_2(0,0) = 1$$

Since $(\partial h_1/\partial x)_{0,0}$ and the similar derivatives are finite, we have, in the vicinity of the origin,

$$\varkappa = \lambda = 0$$

Furthermore, if the approaching upstream flow is irrotational, it can be shown that there is an orientation of the x and y axis for which, in the vicinity of the origin,

$$U = Ax + \ldots, \qquad V = By + \ldots$$

The desired orientation of the x and y axes is along the principal directions of curvature of the surface. To be specific, we may take the x axis along the direction of maximum (convex outward) curvature. Then we shall have $A > 0$ and $A > B$.

If B is positive, we speak of a nodal attachment point; if it is negative, we designate it as a saddle point of attachment.

We see that the given U and V correspond to

$$m = s = 1, \qquad n = r = 0$$

All the parameters on the left-hand side of Eqs. (1) and (2), including the new parameter

$$C \equiv \frac{B}{A} = \frac{Vx}{Uy}$$

are constants. We exclude spinning of the coordinate system, so $\omega_3 = 0$.

We can therefore expect to find similar solutions for f and g, for which $\partial f/\partial x = \partial g/\partial x = 0$. The governing equations reduce to

$$f''' + (f + Cg)f'' + 1 - f'^2 = 0$$

$$g''' + (f + Cg)g'' + C(1 - g'^2) = 0$$

These have been solved numerically by Howarth (1951) for $0 \leq C \leq 1$, and by Davey (1961) for $-1 \leq C \leq 0$. Extension to compressible flow has been made by Poots (1965) for $-.5 \leq C \leq 1$.

The limiting cases $C = 0$ and $C = 1$ clearly correspond to the two-dimensional and axisymmetric stagnation points, respectively.

We have mentioned the practical importance of the nodal solutions in Section III; tabulation and further discussion are given on pp. 461 to 467 of L.B.L. The saddle-point solutions are particularly interesting, in that for $0 > C > -0.4294$ they propose a locally determined solution for a region of flow which lies within the domain of dependence of initial data curves surrounding neighboring nodal points. This is not permissible in principle, but may be approximately useful in practice if the boundary layer has effectively "forgotten" its initial data by the time it gets close to the saddle point. For $-0.4294 > C > -1$, Davey's solutions are particularly intriguing, showing a streamline pattern which changes over from saddle- to node-like behavior as z decreases towards zero. The g' profile shows reverse

flow near the wall and the flow thus appears in a sense to be separated. (A careful discussion of separation follows in a later section.) Nevertheless, for $C > -1$, the displacement thickness of the boundary layer is finite and the external flow is still approaching the wall along the stagnation streamline. The saddle-point region for $-0.4294 > C > -1$ does not lie within the domain of dependence of neighboring nodal points, and in many ways seems to be a region wetted not directly by the flow from upstream, but by a very simple type of wake flow.

"DOUBLE" FALKNER-SKAN FLOWS OVER DEVELOPABLE SURFACES

Developable surfaces can be "unrolled" into a plane without wrinkling, and hence our (ξ,η) coordinates on such surfaces can be everywhere rectangular. This makes

$$h_1 = h_2 = 1, \qquad \kappa = \lambda = 0 \text{ everywhere}$$

Now if $\omega_3 = 0$ (no spinning) and if we can find an orientatio[illegible]d y axes such that

$$U = Ax^m y^n \qquad \text{and} \qquad V = Bx^r y^s$$

with constant A, B, m, n, r, and s, and if the constants are related by

$$r = m - 1 \qquad \text{and} \qquad s = n + 1$$

there will exist similar solutions governed by

$$f''' + \frac{m+1}{2} ff'' + m(1 - f'^2) + \frac{B}{A}\left\{\frac{n+2}{2} gf'' + n(1 - g'f')\right\} = 0$$

and

$$g''' + \frac{m+1}{2} fg'' + (m-1)(1 - f'g') + \frac{B}{A}\left\{\frac{n+2}{2} gg'' + (n+1)(1 - g'^2)\right\} = 0$$

These have been solved by Yohner and Hansen (1958) for $B/A = 1$ and 2, and for many combinations of m and n. The assumed inviscid flows are in general quite rotational, and the resulting nonuniformities of total pressure lead in some cases to "overshooting" f'-profiles.

Similar solutions of a particularly interesting type are also obtained for arbitrary r and m, providing that $n = s = 0$. The inviscid flows are again rotational for $r > 0$, and the governing equations are

$$f''' + \frac{m+1}{2} ff'' + m(1 - f'^2) = 0 \qquad \text{(The Falkner-Skan equation)}$$

and

$$g''' + \frac{m+1}{2} fg'' + r(1 - f'g') = 0$$

These have been solved by Hansen and Herzig (1956) for $m = 0$ and for integer values of r from 1 to 10. These authors noticed that the equation for f' does not contain g', and that the equation for g' is linear, when f' is considered to be known. Thus solutions for any fixed m, but with V given by a polynomial in x, can be generated by superposition. They give examples from the field of turbomachinery and exhibit some interesting comparisons with experiment. Yohner and Hansen (1958) also consider this case, and obtain solutions for all combinations obtainable from $m = 0, 1, 2, 4, 6, 8, 10$ and $r = 0, 1, 2, 4, 6, 8, 10$.

THE INDEPENDENCE PRINCIPLE

The equations treated by Hansen and Herzig (1956) exhibit the special feature that f can be found independently of g, and we may inquire as to how generally this can be done.

We examine Eq. (1) and observe that g appears only in terms containing derivatives with respect to y (including the parameters λ, n, and s), the curvature parameter κ, and the coordinate rotation speed ω_3. Thus, in a y-independent flow over a nonspinning body with a developable surface, f is independent of g. This is the independence principle.

A sample flow would be that over a yawed, infinite cylinder, with y measured along the generators of the cylinder. Then V is a constant and the governing equations are

$$f''' + \left(\frac{m+1}{2}\right) ff'' + m(1 - f'^2) = x\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right)$$

$$g''' + \left(\frac{m+1}{2}\right) fg'' = x\left(f' \frac{\partial g'}{\partial x} - g'' \frac{\partial f}{\partial x}\right)$$

The equation for f is that solved by Smith and Clutter (1963a) for a variety of x-wise pressure distributions. The equation for g is a second-order linear equation for g', so that the generation of solutions for the spanwise flow (g') for any of Smith and Clutter's chordwise flows would be a relatively simple matter. A procedure for doing this numerically is suggested by Lindfield, Pinsent, and Pinsent.*

For the special case m = constant, the equation for f is the Falkner-Skan equation. The corresponding similar span-wise profiles, g', have been found by J. C. Cooke (1950), whose results are tabulated on p. 471 of L.B.L. (In that table they are called g.) Clearly the case m = 1 gives the same f and g as Howarth's stagnation point solution for C = 0 (the cylindrical stagnation point).

CONDITIONS FOR ZERO SECONDARY FLOW

With secondary flow defined as flow at right angles to the inviscid streamlines, we see that it is given by the function G' of Eqs. (3) and (4). (Recall that these are written for "intrinsic" coordinates, with ξ measured along, and η normal to, the inviscid streamlines.)

In Eq. (4), every term contains G or a derivative of G, except for one term, proportional to λ, and another proportional to ω_3. We rule out the second by postulating zero spin. In order to make λ vanish, we first recall that

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*Boundary Layer and Flow Control, G. V. Lachmann, ed., Pergamon Press, 1961 (hereafter cited in the text as B.L.F.C.).

$$\lambda \equiv \frac{\partial(\ell n\ h_1)}{\partial(\ell n\ y)} = h_2 y \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial y} = h_2 \eta \kappa_1$$

where κ_1 is the geodesic curvature of the coordinate line of constant η (in this case the inviscid surface streamline). Geodesic curvature of a surface curve at a point P is the curvature of the projection of that curve on the plane tangent to the surface at P. Surface curves are called geodesics if they have no geodesic curvature and hence appear locally straight when viewed along a normal to the surface at P. There are an infinite number of geodesic curves through P, each with its own tangent direction. (A familiar example is given by the "great circle" routes over the earth's surface from one city to various others.)

Hence, if $\omega_3 = 0$ and if the inviscid flow surface streamlines are geodesic curves of the body (as they are, for example, in the Newtonian hypersonic flow theory) the equation for G' will contain no term independent of G'. Since G' is subject to homogeneous boundary conditions, $G' = 0$ is the indicated solution and there will be no secondary flow.

SIMILAR SOLUTIONS FOR SIMPLE SPINNING FLOWS

When $\omega_3 \neq 0$ it is still possible to find some exact similar boundary-layer solutions, but we omit discussion of these here, for lack of time and space. The reader is referred particularly to Moore (1956) and Mager (1963) for excellent reviews.

VI. CALCULATION SCHEMES FOR ARBITRARY BODY SHAPES

In problems in which the right-hand sides of Eqs. (3) and (4) do not vanish, we must deal with partial differential equations in three independent variables. The usual approach is to replace these equations by a sequence of ordinary differential equations (with independent variable z) by either (1) postulating convergent expansions of f' and G' as power series in x and y, with coefficient functions $f'_{ij}(z)$, $G'_{ij}(z)$ determined by ordinary differential equations (a generalization of the Blasius series approach to two-dimensional problems), or (2) approximating x- and y-derivatives by finite differences, and solving ordinary differential equations at each mesh point (x_i, y_i), (a generalization of Smith and Clutter's approach to two-dimensional problems). In either procedure, careful attention must be paid to the concepts of region of influence and domain of dependence in order to facilitate the organization of computations and to avoid fundamental errors.

SERIES EXPANSIONS

I am not aware of any published study of the application of the Blasius series method to general three-dimensional boundary-layer problems. There have been several applications to the computation of cross-flow velocity profiles in cases which are governed by the independence principle, or which are reduced to quasi-two-dimensional problems by assumption of weak cross flow. Some of these are reviewed by Mager. It appears in some of these works that more terms in the x-expansions of Blasius are needed to provide satisfactorily accurate cross-flow profiles than are needed to get good streamwise profiles. It might be worthwhile to look into the possibilities of applying this method's self-similar nodal attachment point solutions to some body such as an ellipsoid.

FINITE-DIFFERENCE PROCEDURES

Raetz has been the principal contributor to the technique of direct numerical assault on the three-dimensional laminar boundary-layer

equations. He and his co-workers have computed the growth and stability of laminar boundary layers on airplanes, accounting for compressibility and distributed suction. We shall not describe the details of his method, but only sketch here a generalization of the approach applied successfully to nonsimilar two-dimensional boundary layers by Smith and Clutter (1963 and 1964). This method employs the boundary-layer equations in the same transformation as we use here, and seems a little easier to explain. Given the difficulties of guaranteeing the accuracy of finite-difference solutions of partial differential equations, it might actually be worthwhile to develop this procedure into a working alternative to Raetz's method, and to compare the two by applying them to identical problems.

For two-dimensional or axisymmetric incompressible flow, Smith and Clutter (1963a) integrated the equation

$$f''' + \left(\frac{m+1}{2} + \varkappa\right) ff'' + m(1 - f'^2) = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right)$$

for specified distributions of m and $\varkappa$ versus x. The calculation started with initial data given by the appropriate self-similar solution at $x = 0$, and thereafter approximated $\partial f'/\partial x$ and $\partial f/\partial x$ by trailing finite differences. At each discrete value of x_n, an ordinary differential equation for $f(z,x_n)$ (in which the values of f, f' etc. at x_{n-1} appear as variable coefficients) is integrated by numerical methods familiar from the study of similar boundary layers. This constitutes a computationally stable, implicit finite-difference scheme, the accuracy of which could be controlled by restricting the permissible values of $x_n/\Delta x_n$. In their second paper, the method is extended to compressible flow, requiring the simultaneous numerical solution of two coupled ordinary differential equations at each x-value.

The method suggested here is analogous to the former in that finite-difference operations would be used for both $\partial/\partial x$ and $\partial/\partial y$, and a coupled pair of ordinary differential equations integrated at each (x,y) mesh point. Because the finite-differencing must not be done in two dimensions, even more care and study would probably be required to insure computational stability.

The finite-difference procedure would be started at an "initial data surface" surrounding a nodal attachment point or a sharp edge of attachment, and would probably employ intrinsic surface coordinates [Eqs. (3) and (4)].

One point which might require special care in the finite-differencing with respect to y is related to the fact that G' can change sign in the boundary layer. Since the sign of G' indicates the local direction of convection by the secondary flow, and since a fluid particle carries its initial data downstream, it would probably be necessary to devise a differencing scheme which uses the data from the side from which the flow approaches the point in question. If the inviscid flow is irrotational, we can conveniently let the ζ = constant curves be equipotentials, specifically setting $\xi = \Phi(\zeta = 0)/U_\infty$, where U_∞ is a constant reference speed. Then $h_1 = U_\infty/U$ and $\lambda + n = 0$. The determination of h_2 is then directly coupled to the preliminary task of computing the inviscid flow over the surface.* The coordinate curves of constant η (the inviscid streamlines) might rationally be spaced (and h_2 thereby defined) so that

$$\int_{\eta_i}^{\eta_{i+1}} U h_2 \, d\eta$$

is independent of i, when the integral is evaluated along an equipotential. This subdivides the inviscid surface flow which "issues from the attachment point" into equal amounts for equal increment of η. For later reference we note that in this coordinate system, $\varkappa$ measures the lateral spreading of inviscid surface streamlines--a three-dimensional effect which can be artificially eliminated by a generalized Mangler transformation. The combination $m + \varkappa$ measures the surface divergence of the inviscid velocity field, and vanishes for problems in which the inviscid flow is two-dimensional in planes parallel to the (necessarily plane) wall on which the boundary layer grows. Such might be the case on the floor of a wind-tunnel turning section.

*An important modern reference on this subject is Hess and Smith (1963).

WEAK CROSS-FLOW METHODS: PREVALENCE PRINCIPLE

A great many approximate calculations of three-dimensional boundary layers take advantage of the fact that the inviscid streamlines may not deviate too much from surface geodesics, particularly in regions of favorable streamwise pressure gradient, and of course, in the vicinity of special lines of body symmetry. In such cases secondary flows are weak and have small effect on the primary or streamwise velocity profiles. This is called the "principe de prévalence" by Eichelbrenner (1957), and the corresponding theory might be more prosaically termed the "weak cross-flow theory."

In its "zeroth approximation," the weak cross-flow theory sets G' and G equal to zero in Eq. (3), which thus is returned almost to the form treated by Smith and Clutter (1963a), namely

$$f''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) ff'' + \frac{m}{h_1}(1 - f'^2) = \frac{x}{h_1}\left(f' \frac{\partial f'}{\partial x} - f'' \frac{\partial f}{\partial x}\right)$$

This equation can be integrated along each inviscid streamline. The first nonzero estimate of secondary velocities is then obtained from Eq. (4), in which only terms linear in G and its derivatives are retained. This is again an equation which can be integrated along inviscid streamlines by Smith and Clutter's method,

$$G''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) fG'' - \left(\frac{m+\kappa}{h_1}\right) f'G' - \frac{\lambda x}{h_2 y}(1 - f'^2)$$

$$+ 2\left(\frac{\omega_3 L}{U}\right) x (1 - f') = \frac{x}{h_1}\left(f' \frac{\partial G'}{\partial x} - G'' \frac{\partial f}{\partial x}\right)$$

in which f and f' are taken to be the zeroth-order streamwise solutions. The y-derivatives which would appear in the equation for the next approximation to f are then approximated by finite differences constructed as known functions of x by use of the lower approximations to f and G on neighboring streamlines (curves of constant y), and once again we obtain an equation which can be integrated, in the manner of

Smith and Clutter, along the inviscid streamlines. An analogous procedure can be applied to the momentum-integral equations, to convert them from partial differential equations in x and y to a sequence of approximate ordinary differential equations in x. This procedure is discussed by Eichelbrenner and by Mager (1964).

VII. VORTICITY OF THREE-DIMENSIONAL BOUNDARY LAYERS

Sometimes we can establish an intuitive grasp of complicated flow phenomena by considering the sources and history of vorticity, particularly in the case of three-dimensional flows. Our building blocks are the vorticity equation and certain results about sources of vorticity at solid walls, and our end product is an understanding of the origins and development of secondary vorticity and of the pattern of vortex lines and skin friction lines on the wall.

From the vorticity equation we learn that there is only one initial volume source of vorticity in an initially irrotational flow under the action of body forces derivable from a potential. This source proportional to grad ρ X grad p is called baroclinicity by meteorologists, and it vanishes in a fluid of constant density. In its absence, vorticity must initially appear at a boundary of the fluid region. Once in the fluid, it may be diffused from particle to particle by viscous torques, the direction of diffusion being along the normal to the wall in boundary-layer flows. Aside from the effects of baroclinicity and viscous torques, vorticity is simply carried along with the moving fluid in such a way as to conserve the flux of $\vec{\Omega} + 2\vec{\omega}$ across any surface element of fixed material identity. Many implications of this are exhibited in the films "Vorticity" and "Secondary Flow" in the series sponsored by the National Committee for Fluid Mechanics Films.

THE SURFACE SOURCE OF VORTICITY IN INCOMPRESSIBLE FLOW

The diffusive flux of vorticity $\vec{\Omega}$ across a surface in incompressible flow can be described by the tensor ν grad $\vec{\Omega}$, in the sense that the vector $-\vec{n}(\nu \text{ grad } \vec{\Omega})$ gives the rate that vorticity is diffused across a surface normal to the unit vector $\vec{n}$, into the region into which $\vec{n}$ points. If the surface happens to be a solid wall, it is natural to speak of $-\vec{n}(\nu \text{ grad } \vec{\Omega})$ as the vorticity source strength at a point on the wall. Under the assumption of zero slip, and with the assumptions of boundary-layer theory, specifically that $h_1 = h_1(\xi,\eta)$, $h_2 = h_2(\xi,\eta)$, and $h_3 = 1$, we find

$$-\left[\vec{n}(\nu \text{ grad } \vec{\Omega})\right]_{\zeta=0} = \left[\vec{n} \times \nu \text{ curl } \vec{\Omega}\right]_{\zeta=0} + \vec{n}\left[\frac{\partial \Omega_3}{\partial \zeta}\right]_{\zeta=0}$$

From the momentum equation, evaluated at the wall with a no-slip condition, we have

$$\left[\nu \text{ curl } \vec{\Omega}\right]_{\zeta=0} = -\frac{1}{\rho} \text{ grad } P$$

where

$$P \equiv p + \rho\left(\Lambda - \frac{1}{2}\omega^2 R^2\right)$$

The surface source of tangential components of vorticity is predetermined by the inviscid flow over the body. If we imagine the function $P(\xi,\eta)$ to define a pressure or potential "hill," tangential vorticity is fed into the boundary layer at a rate proportional to the slope of this hill, and with the sense of rotation which would be taken by a ball released from rest to start rolling down the hill.

There may also be a surface source of normal vorticity, but this is not predetermined by the inviscid flow. Like the normal component of velocity at the outer edge of a boundary layer, or like the surface values of tangential vorticity, it is part of the response of the boundary layer to imposed conditions.

If we define "primary" or "lateral" vorticity as that component of $\vec{\Omega}$ which is tangential to the wall and normal to the surface streamlines of inviscid flow, and "secondary" or "streamwise" vorticity as that component parallel to the inviscid surface flow, and adopt intrinsic surface coordinates in which $V = 0$, we find that the wall source strength of primary vorticity is $-(U/h_1)\ \partial U/\partial \xi$ and that the wall source strength of secondary vorticity is $\kappa_1 U^2 - 2\omega_3 U$ where κ_1 is the geodesic curvature of the inviscid surface streamlines.

Of course, these expressions give only the tangential components of acceleration of the inviscid flow, along and normal to its surface

streamlines. If the flow were unsteady, there would be a wall source of primary vorticity proportional to $-\partial U/\partial t$.

Lighthill, in L.B.L., pp. 83-84, gives an interesting heuristic derivation of these results without making reference to the pressure field, thereby making clear that, with ρ = constant, it is the acceleration of the inviscid flow that is responsible for the wall sources of vorticity.

THE SURFACE VORTICITY

What we really wish to know in a typical boundary-layer problem is the distribution of surface vorticity which results from the predetermined distribution of wall sources. According to our discussion in Section III, we can see that the surface vorticity at a point $P(\xi,\eta,0)$ must depend upon the sources distributed throughout its "upstream-opening" region of influence. This region is defined by the normal projection on the wall of all the upstream trajectories of particles which arrive in the boundary layer over P. Actually, not all such sources have equal influence on the surface vorticity at P, since the contribution of a given upstream source is proportional to its strength and to a damping factor $(\nu t)^{-1/2}$, where t is an "average time of convection" between the source point and P.

SURFACE VORTEX LINES AND SKIN FRICTION LINES

We can imagine the body surface to be covered with a set of curves everywhere tangent to the surface vorticity vector. Since on a stationary surface with zero slip, the Navier-Stokes stress relationship gives a surface shear stress (skin friction) vector related to the surface vorticity by

$$\vec{\tau}_w = \vec{n}\underline{\underline{T}} = \mu\vec{\Omega} \times \vec{n}$$

we see that the family of skin friction lines, which are everywhere tangent to $\vec{\tau}_w$, form an orthogonal mesh with the surface vortex lines.

The skin friction lines are also frequently referred to as "surface streamlines," since they have, according to the boundary-layer approximations, a tangent which coincides with the limiting direction of boundary-layer streamlines, as $\zeta \to 0$. That is

$$\lim_{\zeta \to 0} \frac{v(\xi,\eta,\zeta)}{u(\xi,\eta,\zeta)} = \frac{0}{0}$$

but the indeterminacy can be resolved, except at isolated singular points, by L'Hospital's Rule, to give

$$\lim_{\zeta \to 0} \frac{v(\xi,\eta,\zeta)}{u(\xi,\eta,\zeta)} = \frac{(\partial v/\partial\zeta)(\xi,\eta,0)}{(\partial u/\partial\zeta)(\xi,\eta,0)} = \frac{\tau_2}{\tau_1}$$

By use of div $\vec{\Omega} = 0$ and $\vec{\tau}_w = \mu(\vec{\Omega} \times \vec{n})_{\zeta=0}$ we can show that

$$\left(\frac{\partial\Omega_3}{\partial\zeta}\right)_{\zeta=0} = \frac{1}{\mu}\,\mathrm{curl}\,\vec{\tau}_w$$

so that the wall source of normal vorticity is proportional to the circulation of the skin friction field around the point of observation.

VIII. TOPOLOGY OF THE SKIN FRICTION LINES

The skin friction lines of a steady flow, like streamlines in the body of the fluid, can run together or cross one another only at isolated singular points, which arise where the vector skin friction vanishes so that τ_2/τ_1 is in turn indeterminate. Such "stagnation points" of the skin friction field are points of flow attachment or separation if at such a point the surface divergence

$$\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial \xi}(h_2\tau_1) + \frac{\partial}{\partial \eta}(h_1\tau_2)\right)$$

is positive or negative, respectively. Such points may also be singular for the surface vorticity field, in which "sources" correspond to points of attachment of vortex filaments.

The topology of the net of skin friction lines and vortex lines is dominated by these singular points, and special studies of the flow neighboring such a point have been made by R. Legendre (1955 and 1956) and by K. Oswatitsch (1957). We give here a very brief resumé of the article by Oswatitsch, which contains many perspective drawings which greatly facilitate understanding.

For a local study near a surface point, we place the origin of coordinates at that point ($\xi = \eta = \zeta = 0$) and adjust the (arbitrary) spacing of the ξ- and η-coordinate surfaces so that $h_1(0,0,0) = h_2(0,0,0) = 1$. Taylor series to the second order in ξ, η, and ζ are postulated for u, v, and w.* These are indicated by the single vector equation

*Many English authors have explored the possibilities of expansion other than the simple Taylor series in the vicinity of separation. For a recent example of this line of thought, see Brown (1965).

$$\vec{u}(\xi,\eta,\zeta) = \vec{u}(0,0,0) + \xi \frac{\partial \vec{u}}{\partial \xi} + \eta \frac{\partial \vec{u}}{\partial \eta} + \zeta \frac{\partial \vec{u}}{\partial \zeta}$$

$$+ \frac{1}{2}\left(\xi^2 \frac{\partial^2 \vec{u}}{\partial \xi^2} + \eta^2 \frac{\partial^2 \vec{u}}{\partial \eta^2} + \zeta^2 \frac{\partial^2 \vec{u}}{\partial \zeta^2}\right)$$

$$+ \left(\xi\eta \frac{\partial^2 \vec{u}}{\partial \xi \partial \eta} + \eta\zeta \frac{\partial^2 \vec{u}}{\partial \eta \partial \zeta} + \zeta\xi \frac{\partial^2 \vec{u}}{\partial \zeta \partial \xi}\right) + \ldots$$

where all the derivatives of $\vec{u}$ are evaluated at the origin (0,0,0). The wall is stationary and impervious, so $\vec{u}(\xi,\eta,0) = 0$. Thus, at points on the wall

$$\frac{\partial \vec{u}}{\partial \xi} = \frac{\partial \vec{u}}{\partial \eta} = \frac{\partial^2 \vec{u}}{\partial \xi^2} = \frac{\partial^2 \vec{u}}{\partial \xi \partial \eta} = \frac{\partial^2 u}{\partial \eta^2} = 0$$

From the continuity equation, we next find that, on the wall

$$\frac{\partial w}{\partial \zeta} = \frac{\partial^2 w}{\partial \xi \partial \zeta} = \frac{\partial^2 w}{\partial \eta \partial \zeta} = 0$$

Finally, we differentiate the continuity equation with respect to ζ and find that, on the wall

$$\frac{\partial^2 w}{\partial \zeta^2} = -\left\{\varkappa_2 \frac{\partial u}{\partial \zeta} + \frac{\partial^2 u}{\partial \xi \partial \zeta} + \varkappa_1 \frac{\partial v}{\partial \zeta} + \frac{\partial^2 v}{\partial \eta \partial \zeta}\right\}$$

where $\varkappa_1$, $\varkappa_2$ are the geodesic curvatures of the surface coordinate mesh.

The momentum equation can be used to get

$$\mu \frac{\partial^2 u}{\partial \zeta^2} = -\left(\frac{\partial \mu}{\partial \zeta}\frac{\partial u}{\partial \zeta}\right) + \frac{\partial p}{\partial \xi}$$

and

$$\mu \frac{\partial^2 v}{\partial \zeta^2} = -\left(\frac{\partial \mu}{\partial \zeta}\frac{\partial v}{\partial \zeta}\right) + \frac{\partial P}{\partial \eta}$$

for points on the wall, where $P \equiv [p + \rho(\Lambda - 1/2\ \omega^2 R^2)]$. Finally, we introduce the skin friction vector

$$\vec{\tau}_w = \mu \frac{\partial \vec{u}}{\partial \zeta}(\xi,\eta,0)$$

and specify that the origin (0,0,0) is a singular point at which $\vec{\tau}_w = 0$. All this leads to the scalar expressions

$$\mu u(\xi,\eta,\zeta) = \frac{\zeta^2}{2}\frac{\partial P}{\partial \xi} + \eta\zeta \frac{\partial \tau_1}{\partial \eta} + \xi\zeta \frac{\partial \tau_1}{\partial \xi} + \ldots$$

$$\mu v(\xi,\eta,\zeta) = \frac{\zeta^2}{2}\frac{\partial P}{\partial \eta} + \eta\zeta \frac{\partial \tau_2}{\partial \eta} + \xi\zeta \frac{\partial \tau_2}{\partial \eta} + \ldots$$

$$\mu w(\xi,\eta,\zeta) = -\frac{\zeta^2}{2}\left(\frac{\partial \tau_1}{\partial \xi} + \frac{\partial \tau_2}{\partial \eta}\right)$$

in which all the differential coefficients are evaluated at (0,0,0).

From these equations follow the differential equations of the streamlines in the vicinity of (0,0,0), namely,

$$\frac{d\eta}{d\xi} = \frac{v}{u} = \frac{\zeta \frac{\partial P}{\partial \eta} + 2\eta \frac{\partial \tau_2}{\partial \eta} + 2\xi \frac{\partial \tau_2}{\partial \xi}}{\zeta \frac{\partial P}{\partial \xi} + 2\eta \frac{\partial \tau_1}{\partial \eta} + 2\xi \frac{\partial \tau_1}{\partial \xi}}$$

$$\frac{d\zeta}{d\xi} = \frac{w}{u} = -\frac{\zeta\left(\frac{\partial \tau_1}{\partial \xi} + \frac{\partial \tau_2}{\partial \eta}\right)}{\zeta \frac{\partial P}{\partial \xi} + 2\eta \frac{\partial \tau_1}{\partial \eta} + 2\xi \frac{\partial \tau_2}{\partial \xi}}$$

$$\frac{d\zeta}{d\eta} = \frac{w}{v} = - \frac{\zeta\left(\frac{\partial\tau_1}{\partial\xi} + \frac{\partial\tau_2}{\partial\eta}\right)}{\zeta\frac{\partial p}{\partial\eta} + 2\eta\frac{\partial\tau_2}{\partial\eta} + 2\xi\frac{\partial\tau_2}{\partial\xi}}$$

Oswatitsch gives the general analytic solutions for these equations, from which we sample only a few useful results. The first of these has to do with the skin friction lines themselves, whose trajectories (at $\zeta = 0$) are governed by the nonlinear equation

$$\frac{d\eta}{d\xi} = \frac{\eta\frac{\partial\tau_2}{\partial\eta} + \xi\frac{\partial\tau_2}{\partial\xi}}{\eta\frac{\partial\tau_1}{\partial\eta} + \xi\frac{\partial\tau_1}{\partial\xi}}$$

We first see whether any skin friction lines enter the origin by seeking the singular straight-line solutions

$$\frac{d\eta}{d\xi} = \frac{\eta}{\xi} = \tan\theta$$

These exist provided that

$$\tan\theta = \frac{\frac{\partial\tau_2}{\partial\eta} - \frac{\partial\tau_1}{\partial\xi} \pm \left[\left(\frac{\partial\tau_2}{\partial\eta} - \frac{\partial\tau_1}{\partial\xi}\right)^2 - 4\left(\frac{\partial\tau_1}{\partial\eta}\frac{\partial\tau_2}{\partial\xi}\right)\right]^{1/2}}{2\frac{\partial\tau_1}{\partial\eta}}$$

This has two distinct real values if

$$\left(\frac{\partial\tau_2}{\partial\eta} - \frac{\partial\tau_1}{\partial\xi}\right)^2 - 4\left(\frac{\partial\tau_1}{\partial\eta}\frac{\partial\tau_2}{\partial\xi}\right) = \Delta^2 - 4J > 0$$

where

$$\Delta \equiv \frac{\partial\tau_1}{\partial\xi} + \frac{\partial\tau_2}{\partial\eta}, \qquad J \equiv \frac{\partial\tau_1}{\partial\xi}\frac{\partial\tau_2}{\partial\eta} - \frac{\partial\tau_1}{\partial\eta}\frac{\partial\tau_2}{\partial\xi}$$

These parameters, together with

$$\Omega \equiv \frac{\partial \tau_1}{\partial \eta} - \frac{\partial \tau_2}{\partial \xi}$$

completely determine the skin friction field near the singular point. There are few general physical restrictions on the values of the divergence ($\cdot$) or the curl ($-\Omega$), which may develop in the skin friction field, so we may expect to encounter in practice singular points of great variety.

SYMMETRIC SINGULAR POINTS

The skin friction field may be curl-free ($\Omega = 0$) everywhere in very special cases (e.g., axisymmetric flow, potential inviscid flow with geodesic surface streamlines), along lines of symmetry on more generally shaped bodies (e.g., an ellipsoid of revolution at angle of attack), or at least at points of attachment of externally irrotational flow. Singular points encountered under these conditions exhibit a symmetry, in that the singular trajectories through them are mutually perpendicular, so that the ξ and η axes can be rotated to coincide with them. The symmetry then yields

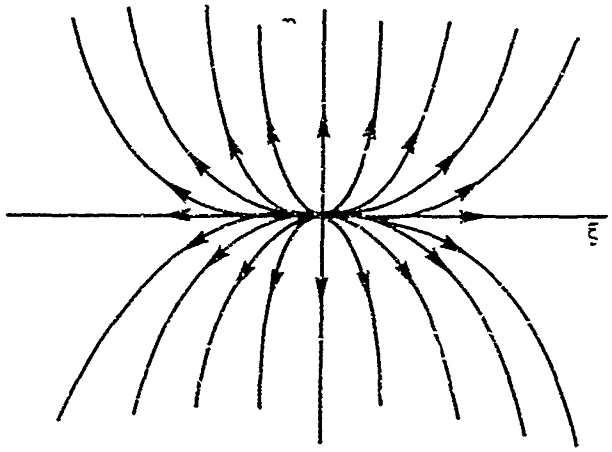
$$\frac{\partial \tau_1}{\partial \eta} = \frac{\partial \tau_2}{\partial \xi} = 0$$

and the general trajectory equation becomes

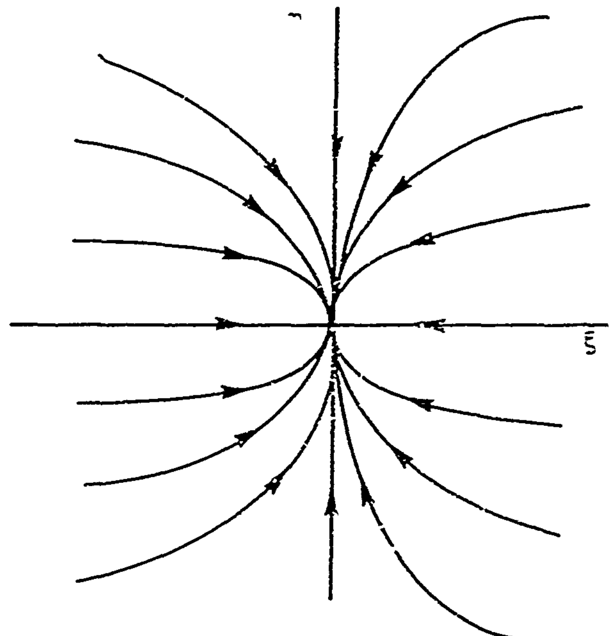
$$\frac{d\eta}{d\xi} = c\,\frac{\eta}{\xi}, \qquad \text{with } c \equiv \left(\frac{\partial \tau_2}{\partial \eta}\right) \Big/ \left(\frac{\partial \tau_1}{\partial \xi}\right)$$

To be consistent with Oswatitsch's terminology, we can call c the convergence of the singular point. As shown in Figs. 2 and 3, the solution curves

$$\eta = a\xi^c$$

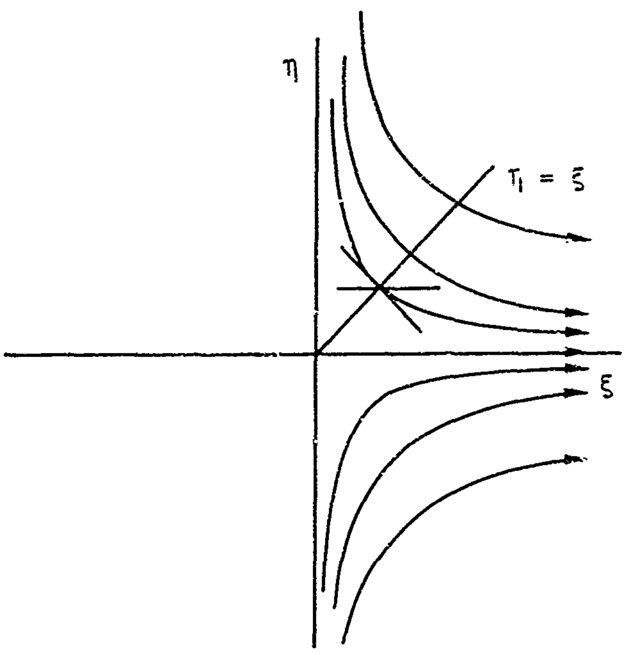


$c > 1, \qquad \Delta > 0$

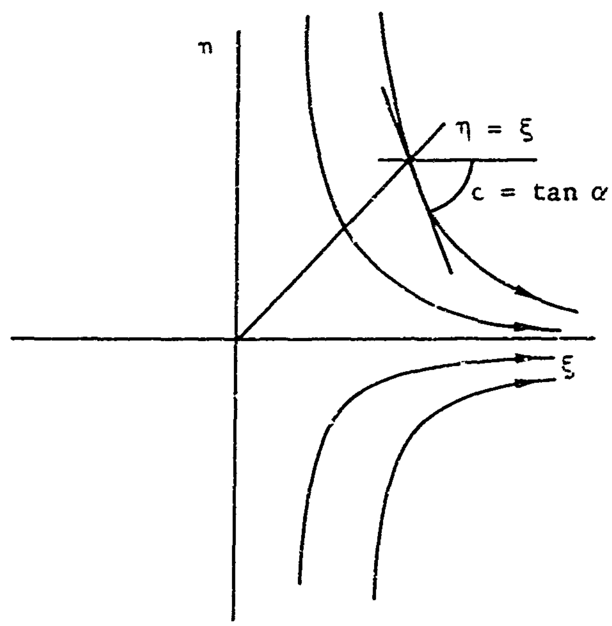


$0 < c < 1, \qquad \Delta < 0$

Fig. 2 -- Symmetric nodes.



$-1 < c < 0, \qquad \Delta > 0$



$c < -1, \qquad \Delta < 0$

Fig. 3 -- Symmetric saddles.

all converge at the origin when $c > 0$ (a node), while all approaching trajectories except two (along the ξ and η axes) diverge away from the origin when $c < 0$ (a saddle point). Convergence, defined in this sense, is unfortunately not the opposite of divergence, Δ, but is related to it by

$$\Delta = \frac{\partial \tau_1}{\partial \xi} (1 + c)$$

From any given trajectory plot, one can quickly evaluate c as the slope of a skin friction curve at the point where it crosses the line $\eta = \xi$. It is easy to tell the sign of Δ at a node by looking at the arrows on the skin friction lines. At a saddle point we can derive the rule of thumb: $\Delta > 0$ (attachment) occurs when the approaching skin friction lines are more closely packed (near their axis of symmetry) than the departing skin friction lines.

ANGLE OF THE ATTACHING OR SEPARATING STREAMLINE

For the symmetric singular points, $\Omega = 0$, it is easy to find the slope, $d\zeta/d\xi$, of the singular streamline which leaves the wall at the singular point. We suppose that $\eta = 0$ is a plane of symmetry, so that $\partial P/\partial \eta = 0$ (hence $d\zeta/d\eta = \infty$).

The general equation is

$$\frac{d\zeta}{d\xi} = - \frac{\zeta\Delta}{\zeta \frac{\partial P}{\partial \xi} + 2\xi \frac{\partial \tau_1}{\partial \xi}}$$

and the singular, straight-line solution, is

$$\frac{d\zeta}{d\xi} = \frac{\zeta}{\xi} = \tan \beta$$

where

$$\tan \beta = -\left(\frac{3 \frac{\partial \tau_1}{\partial \xi}}{\frac{\partial P}{\partial \xi}} \right) (1 + c)$$

The first factor in this expression is the familiar angle of separation of a two-dimensional boundary layer, with specified values of $\partial P/\partial \xi$ (> 0 for separation) and $\partial \tau_1/\partial \xi$ (< 0, but bearing no unique relation to $\partial P/\partial \xi$). We see that at symmetric three-dimensional singular points it is modified by the factor $1 + c$, which is > 1 for nodes, and < 1 for saddles. Note that at attachment points of an irrotational inviscid flow, $\partial P/\partial \xi = 0$, and the attaching streamline is normal to the wall.

MORE GENERAL SINGULAR POINTS

As illustrated by Legendre and Oswatitsch, $J > 0$ generally denotes a node and $J < 0$ a saddle point. When $0 < J < \Delta^2/4$, the friction lines behave qualitatively as around a symmetric node, while for $J > \Delta^2/4$ they spiral into the origin. (Replace Δ^2 by Ω^2 and friction lines by vortex lines, and the last sentence is again true.) If J is greater than both $\Delta^2/4$ and $\Omega^2/4$, both skin friction lines and vortex lines spiral. Evidently spiral detachment ($J > \Delta^2/4$ and $\Delta < 0$) occurs quite commonly in practice, notably on delta wings. Lighthill's discussion and sketches on pp. 74-82 of L.B.L. are very helpful.

NUMBER OF NODES AND SADDLES ON A CLOSED, SIMPLY CONNECTED BODY

Lighthill cites (p. 76, L.B.L.) the following topological law, that the number of nodal points must exceed the number of saddle points by 2. He says "one may argue that the infinity of skin friction lines on the surface must begin and end somewhere, which indicates that there is at least one nodal point of attachment and one nodal point of separation. If there are two nodal points of attachment, the skin friction lines from each must somewhere run into one another, and so have to divide at a saddle point."

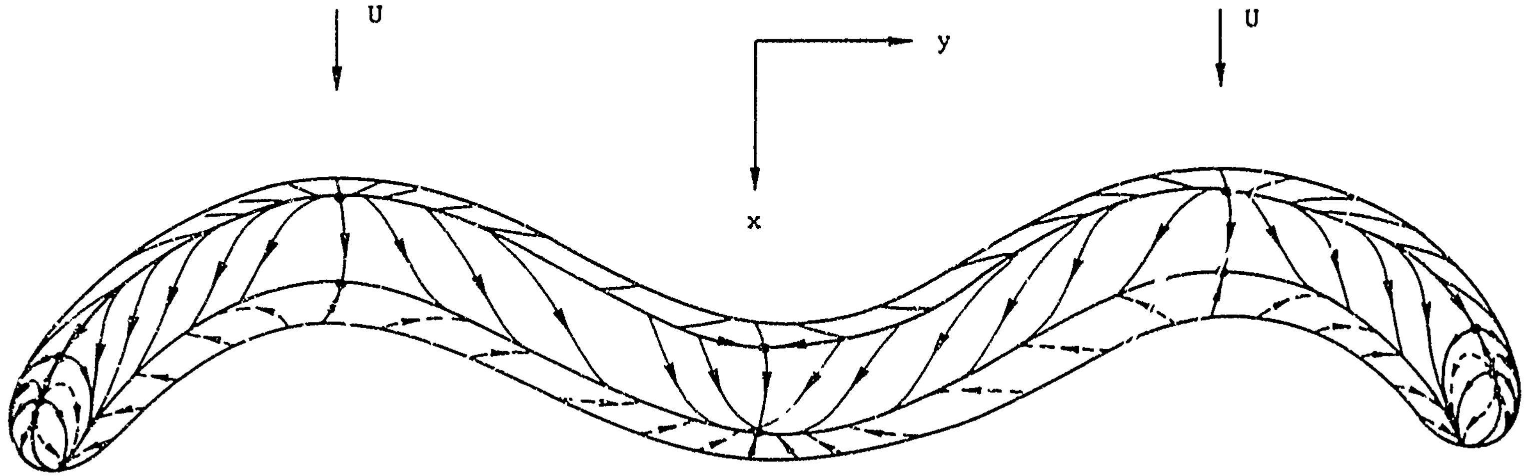
IX. SEPARATION

SEPARATION LINES

From the point of view of topology of skin friction curves, unseparated flow exists when the entire body surface is covered by friction lines which originate at upstream nodal points of attachment, and disappear into an equal number of downstream nodes of separation. In practice, this situation is atypical; experiments ordinarily show the existence of other nodal "reattachment" points, at which the incoming streamline approaches not from the free stream, but from the interior of a wake bubble. The skin friction lines from the upstream attachment node and those from the wake reattachment node run towards each other and hence must divide at a saddle point, which will ordinarily be a saddle point of separation. The singular skin friction lines approaching this saddle point come from the nodes of attachment and reattachment. The singular skin friction lines departing from the saddle point run eventually into a node or two nodes of separation.

The latter singular friction lines separate the body surface into a region which is wetted by the upstream flow, and a region which is wetted by the reattaching wake flow. They are thus logically defined as the separation lines.

The concepts of a separation line and of nodal points of attachment and of reattachment, a considerable variety of symmetric singular points, and the rule regarding the relative number of nodes and saddles are all illustrated in Fig. 4. In it we view a "foot-long hot dog," sagging somewhat in the middle. A uniform flow approaches along the x axis, and the hot dog is supposed to possess mirror symmetry in the planes y = 0 and z = 0. We view it along a line that is inclined slightly upstream and spanwise from the z axis, so that we see the forward attachment points and the interesting details on one end. The sketch is only intended to appear plausible, and does not come from an actual experiment or calculation. In particular, if the hot dog were drooped more in the middle and less at the ends, many more singular points might be expected to appear in those regions. (These guesses derive from the paper on saddle points of attachment by Davey



2 nodes of upstream attachment
4 nodes of wake attachment
4 nodes of separation
1 saddle of upstream attachment
1 saddle of wake attachment
6 saddles of separation

Solid skin friction trajectories denote upstream-wetted region, dashed ones denote wake-wetted regions

Fig. 4 -- The foot-long hot dog.

(referred to in Section VI), and from drawings derived from observed flow on a yawed ellipsoid by Eichelbrenner, and exhibited in the survey article by Cooke and Brebner in B.L.F.C.

INFLUENCE OF SEPARATION ON BOUNDARY-LAYER COMPUTATIONS

Separation limits the applicability of boundary-layer theory in two ways. The first can be anticipated within the framework of the theory and our previous discussion of domains of dependence and appropriately chosen initial data arcs. From these considerations it is clear that we cannot "march" our computations across a separation line, since the flow on the downstream side of such a line falls into the region of influence of the unknown reattaching flow.

The next question is whether we can march up to the separation line from the wake-wetted side as well as from the upstream-wetted side, and thereby compute the boundary-layer development over the entire body. The answer appears to be almost always negative. One difficulty is that we do not know, a priori, where the wake-reattachment nodes will be. They do not in general coincide with any singular features of an irrotational inviscid solution (and in practice they tend to wander temporally, at any Reynolds numbers for which the boundary layers may be expected to be thin). Furthermore, the fluid attaching at these points, unlike the free stream fluid attaching at upstream nodal points, has had a recent experience of viscous action, so that we cannot ordinarily expect to find self-contained similar solutions for the vicinity of the reattachment node. Finally, even if these difficulties and those of determining the inviscid surface velocities in the wake-wetted region could be surmounted, there remains the possibility that near the separation line the direction of dominant viscous diffusion differs significantly from that of the wall normal, thus invalidating our boundary-layer equations locally.

The second adverse effect of separation is its displacement effect, which causes the actual pressure distribution to deviate from that of inviscid theory over a region which may extend significantly upstream of the computed line of separation. This effect would then render inaccurate the boundary-layer computations in this vicinity,

and in particular their prediction of the location of the separation line.

Though little direct evidence as yet exists, there is reason to hope that this latter effect may be less damaging in many types of three-dimensional flow than in two-dimensional flow. These intuitive hopes derive from the expectations that (1) the angle of separation may be quite small at and near the saddle points of separation (Δ being small there); (2) the line of separation may be considerably "swept back" relative to the local inviscid flow; (3) inviscid flow can to some extent go around the regions of vigorous outflow near the nodal separation points.

To set against this optimistic view is the observation that spiral nodes of separation are often situated at the feet of tightly wound and vigorous "trailing vortices" which may induce profound perturbations of the neighboring inviscid fields.

X. MOMENTUM-INTEGRAL METHODS

Granting the fact that the direct numerical integration of Eqs. (1) and (2) or of Eqs. (3) and (4) will be very lengthy even if the method is computationally stable and well-posed, we may wish to investigate approximate methods which satisfy the equations in a (possibly weighted) mean sense at each x and y. These are the momentum-integral methods, of which a particularly comprehensive review is given by Lindfield, Pinsent, and Pinsent in B.L.F.C. We may derive these equations directly from Eqs. (1) and (2) or Eqs. (3) and (4) as follows.

THE MOMENTUM-INTEGRAL EQUATIONS IN INTRINSIC COORDINATES

Holding x and y constant, we integrate Eqs. (3) and (4) over z from 0 to ∞, and employ the appropriate boundary conditions for f and G. For an impervious wall, we have

$$f(0) = f'(0) = G(0) = G'(0) = 0$$

The matching conditions are $f'(\infty) = 1$ (hence $f''(\infty) = 0$, $G'(\infty) = G''(\infty) = 0$). Integration is assisted by the device of setting

$$f'' = -(1 - f')'$$

so that in

$$\int_0^\infty ff''\,dz = [-f(1 - f')]_0^\infty + \int_0^\infty (1 - f')f'\,dz$$

the first term on the right vanishes at both limits.

We introduce the ' ls

$$\delta_1(x,y) \equiv \int_0 (1 - f')\,dz, \qquad \delta_2(x,y) \equiv \int_0^\infty -G'\,dz$$

$$\theta_{11}(x,y) \equiv \int_0^\infty (1 - f')f'\, dz, \qquad \theta_{12}(x,y) \equiv \int_0^\infty (1 - f')G'\, dz$$

$$\theta_{21}(x,y) \equiv \int_0^\infty -G'f'\, dz, \qquad \theta_{22}(x,y) \equiv \int_0^\infty -G'^2\, dz$$

and get

$$-f''(0) + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right)\theta_{11} + \frac{m}{h_1}(\delta_1 + \theta_{11}) + \frac{x}{y}\frac{1}{h_2}\left\{\left(\frac{n}{2} + \lambda\right)\right.$$

$$\left.\theta_{12} + (n + \lambda)(\delta_2 + \theta_{12})\right\} - \frac{\kappa}{h_1}\theta_{22} - 2\frac{\omega_3 L}{\nu}x\delta^2$$

$$= -x\left\{\frac{1}{h_1}\frac{\partial\theta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{12}}{\partial y}\right\}$$

For easy comparison with the equation of Mager (1964), we collect terms in θ_{11} and θ_{12}, and note that

$$\frac{n+\lambda}{h_2 y} = \frac{-\Omega_3 L}{U}$$

where Ω_3 is the ζ-component of the inviscid-flow vorticity. This allows a neat combination of Ω_3 and ω_3 into one term which vanishes in many applications. Then we have

$$\frac{1}{h_1}\frac{\partial\theta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{12}}{\partial y} = \frac{f''(0)}{x} - \frac{1}{h_1 x}\left\{\left(\frac{3m+1}{2} + 2\kappa\right)\theta_{11} - \kappa(\theta_{11} + \theta_{22}) + m\delta_1\right\}$$

$$- \frac{1}{h_2 y}\left\{\left(\frac{3n}{2} + 2\lambda\right)\theta_{12}\right\} + (\Omega_3 + 2\omega_3)\frac{L}{U}\delta_2 \qquad (5)$$

and by a similar calculation on the cross-stream momentum equation

$$\frac{1}{h_1}\frac{\partial\theta_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{22}}{\partial y} = \frac{G''(0)}{x} - \frac{1}{h_1 x}\left\{\left(\frac{3m+1}{2} + 2\kappa\right)\theta_{21}\right\} - (\Omega_3 + 2\omega_3)\frac{L}{U}\delta_1$$

$$- \frac{1}{h_2 y}\left\{\left(\frac{3n}{2} + 2\lambda\right)\theta_{22} - \lambda(\theta_{11} + \theta_{22}) + n\delta_2\right\} \qquad (6)$$

Here we have two coupled first-order partial differential equations for the six integral quantities (δ_1, δ_2, θ_{11}, θ_{12}, θ_{21}, and θ_{22}) and the two wall-derivatives $f''(0)$ and $G''(0)$.

COMPATIBILITY CONDITIONS

We have also at our disposal a variety of algebraic "compatibility conditions" obtained from Eqs. (3) and (4) and from the z-derivatives of these equations, evaluated at the wall ($z = 0$):

First Compatibility Condition

$$f''(0) = -\frac{m}{h_1} \qquad \text{and} \qquad G''(0) = \frac{\lambda x}{h_2 y} - 2\,\frac{\omega_3 L x}{U}$$

Second Compatibility Condition

$$f''(0) = 0 \qquad \text{and} \qquad G'' = 2\left(\frac{\omega_3 L}{U}\right) x f''(0)$$

(In Lindfield, Pinsent, and Pinsent these conditions and the momentum-integral equations are generalized to include the effects of wall suction or blowing.)

SINGLY INFINITE FAMILIES OF VELOCITY PROFILES

We follow Head (1961) (whose excellent survey of integral methods for two-dimensional flows precedes the article by Lindfield et al. in

Vol. 2 of B.L.F.C.) in calling a family of velocity profiles singly infinite if the shape of a member profile, with respect to a linearly stretched z coordinate, is determined by a single parameter or shape factor. The profile, expressed in terms of z itself, then requires two parameters, the shape factor and the scale factor.

Mathematically, we write

$$f'(x,y,z) = \frac{\partial}{\partial z} f\left[\chi(x,y) \quad \frac{z}{\delta(x,y)}\right]$$

$$G'(x,y,z) = \frac{\partial}{\partial z} G\left[\Sigma(x,y) \quad \frac{z}{\Delta(x,y)}\right]$$

where χ and Σ are the shape factors, δ and Δ the scale factors.

If we employ singly infinite profile families for both f' and G', then we shall need four equations to determine the four parameters as functions of x and y. These are at hand in the two momentum-integral equations and either set of compatibility conditions.

Usually we design the functions f' and G' so that the shape factors can be eliminated algebraically by use of the chosen compatibility conditions, whence Eqs. (5) and (6) can be rewritten as a pair of quasi-linear partial differential equations for the scale factors, δ and Δ.

Symbolically (now following Mager's discussion (1964)) we have

$$A \frac{\partial\delta}{\partial x} + B \frac{\partial\delta}{\partial y} + C \frac{\partial\Delta}{\partial x} + D \frac{\partial\Delta}{\partial y} + E = 0 \qquad (7)$$

$$A' \frac{\partial\delta}{\partial x} + B' \frac{\partial\delta}{\partial y} + C' \frac{\partial\Delta}{\partial x} + D' \frac{\partial\Delta}{\partial y} + E' = 0 \qquad (8)$$

where, for example,

$$A(x,y) = \frac{1}{h_1} \frac{\partial\theta_{11}}{\partial\delta}, \qquad A' = \frac{1}{h_1} \frac{\partial\theta_{21}}{\partial\delta}, \text{ etc.}$$

Before beginning a detailed discussion of solution procedures and approximating expressions for F' and G', we note the following gain in simplicity from Eqs. (3) and (4) to Eqs. (5) and (6).

Our proposed solution method for Eqs. (3) and (4) was to employ finite differences in x and y, to obtain two coupled ordinary differential equations at each (x,y) mesh point. These latter were to be solved numerically, subject to the split boundary conditions (at $z = 0$ and $z = \infty$).

If we repeat this procedure for the set (5) and (6), we have, after finite-differencing in x and y, two coupled algebraic equations to be solved at each mesh point. Clearly the computational savings are great, even though the computational effort of the momentum-integral method is considerably greater here than in two-dimensional problems.

WEIGHTED INTEGRAL EQUATION

For an alternative to the compatibility conditions we may use another pair of differential equations to relate the shape factors to the scale factors. Such equations may be constructed by first multiplying the momentum equations (3) and (4) by a "weighting" factor (or factors) and then integrating over z. These new partial differential equations will be coupled to the unweighted momentum-integral equations, and the substitution of them for the compatibility conditions will improve the accuracy of the momentum-integral method only in return for an increased computational effort. Head gives a number of examples, comparing the accuracy of various alternative methods for some important and particularly trying two-dimensional boundary-layer problems.

One's hope for a gain in accuracy through introduction of the weighted momentum-integral equations is strongest in cases involving a sudden change in pressure gradient (either streamwise or cross-stream). In such cases the first compatibility conditions, which always insist on an exactly correct vorticity source strength at the wall, may wrench the rest of the profile severely out of shape in the region just downstream of the sudden change. This effect is worst when we employ singly infinite profile families, which are incapable of the sort of shape adjustments which occur in the real boundary layer

as the new vorticity gradually diffuses out from the wall. The typical symptom of inaccuracy is a discontinuity or excessively rapid change of surface vorticity (skin friction) and displacement thickness at the location of the sudden change of pressure gradient. In these regions the curve of vorticity versus z has the correct slope, but a very inaccurate intercept, at $z = 0$.

The weighted momentum-integral equations ordinarily give special emphasis to the role of convection in the boundary layer, the usual weighting factors being f' for Eq. (3) and G' for Eq. (4). In a region of sudden pressure change these equations require a good description of the quasi-inviscid response of velocities to pressure gradients, and since that response is dominant in the outer regions into which the fresh vorticity has not yet diffused, they give a better picture of the response of "overall" boundary-layer profile shape and thickness. Close to the wall the vorticity is forced to assume reasonably correct values since in a singly infinite profile family the wall vorticity and the boundary layer thickness are usually intimately connected. The slope of the vorticity profile at $z = 0$, and hence the vorticity source strength, will probably be given poorly by this method in the regions of sudden adjustment, but unless great accuracy is needed in the velocity profile curvature, this error may be quite tolerable.

SAMPLE SIMPLY INFINITE PROFILE FAMILIES

Head and Lindfield et al. present good discussions of the way in which profile families can be generated by using polynomials, transcendental functions, or numerical functions of the stretched normal distance. The usual procedure is to make a linear combination of two functions, one of which is selected to give a fair approximation to the profile at attachment points and in the neighboring regions in which the streamwise pressure gradient is favorable and in which the inviscid streamline curvature has the same sign as at attachment. Samples of such "first" functions, from the work of Timman (1951) and Zaat (1956) are

$$f_1' = \operatorname{erf}\phi - \frac{2}{3\sqrt{\pi}}\,\phi e^{-\phi^2}$$

$$\phi \equiv \frac{z}{\delta}$$

which satisfies

$$f_1'(0) = 0, \qquad f_1''(0) = \frac{4}{3\sqrt{\pi}\,\delta}, \qquad f_1'''(0) = 0$$

$$f_1'(\infty) = 1, \qquad f_1' = f_1''' = f_1^{(n)}(\infty) = 0$$

and

$$G_1'(\Psi) = \Psi e^{-\Psi^2}$$

$$\Psi \equiv \frac{z}{\Delta}$$

which gives

$$G_1'(0) = 0, \qquad G_1''(0) = \frac{1}{\Delta}, \qquad G_1'''(0) = 0$$

$$G_1'(\infty) = G_1''(\infty) = G_1^{(n)}(\infty) = 0$$

These functions, which fulfill the boundary conditions on f' and G', are given in Fig. 5. The fact that $f_1'''(0) = G_1'''(0) = 0$ has an important bearing on the algebraic simplicity of the subsequent theory when the first compatibility equations are used.

The second component of the profile must now vanish at $z = 0$ and $z = \infty$, but can otherwise be chosen arbitrarily with the special objective of imitating the profound changes in profile shape which accompany

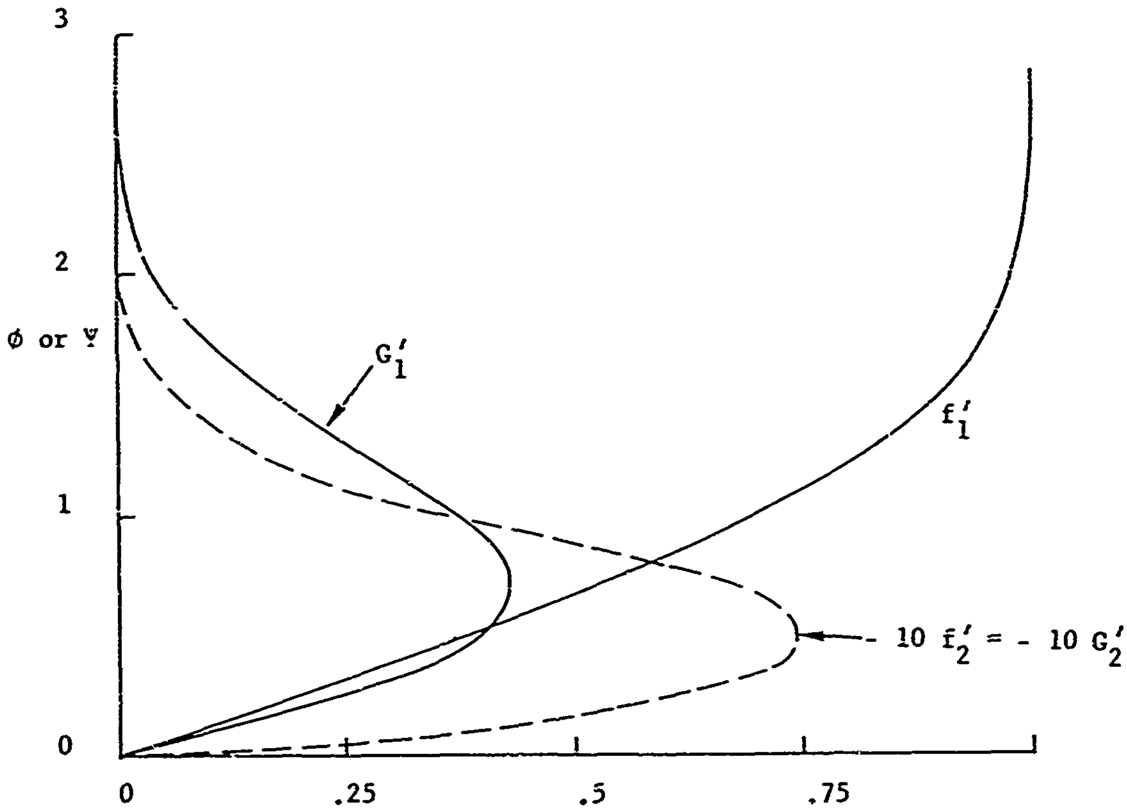


Fig. 5 -- Zaat and Timman's profile functions.

the onset of adverse streamwise pressure gradients and changes in the sign of the inviscid streamline curvature. To achieve this objective in both f' and G', Zaat and Timman propose the function

$$f_2' = \frac{1}{2}\left(\operatorname{erfc}\phi - e^{-\phi^2}\right) + \frac{1}{3\sqrt{\pi}}\,\phi e^{-\phi^2}$$

and let G_2' be this function of Ψ. This gives

$$f_2'(0) = 0, \qquad f_2''(0) = -\frac{2}{3\sqrt{\pi}\,\delta}, \qquad f_2(0) = 1$$

$$f_2'(\infty) = 0, \qquad f_2'' = f_2''' = f_2^{(n)}(\infty) = 0$$

Thus we could write

$$f'(x,y,z) = f_1'\left(\frac{z}{\delta}\right) - \chi f_2'\left(\frac{z}{\delta}\right)$$

and

$$G'(x,y,z) = A_o G_1'\left(\frac{z}{\Delta}\right) - \Sigma\ G_2'\left(\frac{z}{\Delta}\right)$$

It can be readily verified that both of these combinations can describe "two-sided" as well as "one-sided" profiles, as a result of variation in χ and Σ.

The extra coefficient A_o is permissible as far as satisfaction of the boundary conditions of G' is concerned, and may (I think) be employed to get the calculation started in the vicinity of an attachment point. In such a region exact cross-flow profiles are given in terms of the similar solutions of Howarth, being

$$G' = \frac{g' - f'}{\left(\frac{x}{cy} + \frac{cy}{x}\right)}$$

where x, y, c, g' and f' are as defined in Section VI. While the functions g' and f' are universal in the neighborhood of the attachment point, we see that G' will vary with x/y along the initial data arc. Since Zaat and Timman's G_1' has about the same shape as $g' - f'$, we might take $\Sigma = 0$ on the initial data arc, and let $A_o = (x_o/cy_o + cy_o/x_o)^{-1}$.

DOUBLY INFINITE PROFILE FAMILIES

While the singly infinite families of profiles which are composed from Zaat and Timman's functions may assume at least qualitatively reasonable shapes in interesting problems, particularly if we employ the "gentler" but more tedious procedure with four integral equations and ignore the compatibility conditions, they do not have enough flexibility to permit accurate representation of velocity profiles, such as are needed in attempts to predict separation or laminar stability. Head and Lindfield et al. show how this situation may be improved by the

introduction of doubly infinite profile families, which are characterized by two shape factors and a scale factor for each family.

If two new shape factors are introduced, then we need six equations in all. These may be the two momentum-integral equations, two weighted momentum-integral equations and the first compatibility conditions. Obviously other combinations can be used, and Head has tested some of these for two-dimensional flows. If one has decided to use the four integral equations for singly infinite profile families, not too much extra work is involved in going to doubly infinite profile families.

In fact, for three-dimensional boundary layers, one might expect that the f' and G' profiles could possibly "share" some parameters, so that doubly infinite families could be used without the need for more equations. That is, we might be able to extend our profiles to

$$f' = f_1' - \chi f_2' - \Sigma f_3'$$

and

$$G' = A_o G_1' - \Sigma G_2' - \chi G_3'$$

but it would require considerable experience to discover what, if any, expressions for f_3' and G_3' might make this scheme profitable.

SOLUTION OF THE MOMENTUM-INTEGRAL AND ENERGY-INTEGRAL EQUATIONS

Even though the momentum-integral equations can presumably be solved numerically with much less effort than the full boundary-layer equations, they have seldom been applied to truly three-dimensional problems. The exceptional examples seem to be discussed by Zaat (1956) and Eichelbrenner (1957), and both of these employed solution methods adapted only to regions with weak cross flows. Lindfield, Pinsent, and Pinsent (1961) give a general discussion of the use of weighted integral equations, but work no examples in that publica-

tion. Only Timman (1951) and Mager (1957) appear to have studied the characteristics of the partial differential Eqs. (7) and (8), and no study seems to have been made of the characteristics of the fourth-order system of two momentum-integral plus two energy-integral equations. I have made various attempts to fill this gap in this Memorandum, but conclude that the topic is too lengthy and too specific to be worth including.

Derivation of Weighted Momentum-Integral Equations

If we weight Eq. (3) by f' and then integrate over z, the terms on the right-hand side contribute the derivative terms in the streamwise-energy-integral equation. These are

$$-\frac{x}{2}\left(\frac{1}{h_1}\frac{\partial \epsilon_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial \epsilon_{12}}{\partial y}\right)$$

where

$$\epsilon_{11} \equiv \int_0^\infty f'(1 - f'^2)\, dz$$

$$\epsilon_{12} \equiv \int_0^\infty G'(1 - f'^2)\, dz$$

If we try, following Lindfield et al., to weight Eq. (4) also by f', we find it is impossible to extract the differentiations in x and y from under the integral sign. (I can only conclude that Lindfield et al. succeeded in doing this by virtue of an error in calculus.) The resulting equation would thus be unbearably tedious to employ. However, if we weight Eq. (4) by the velocity G', no such difficulty arises. The derivative terms are

$$-\frac{x}{2}\left(\frac{1}{h_1}\frac{\partial \epsilon_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial \epsilon_{22}}{\partial y}\right)$$

where

$$\epsilon_{21} \equiv -\int_0^\infty f' G'^2 \, dz$$

$$\epsilon_{22} \equiv -\int_0^\infty G'^3 \, dz$$

The momentum and energy integral equations may thus be summarized as

$$\frac{1}{h_1}\frac{\partial\theta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{12}}{\partial y} + \text{algebraic terms} = 0$$

$$\frac{1}{h_1}\frac{\partial\theta_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{22}}{\partial y} + \text{algebraic terms} = 0$$

$$\frac{1}{h_1}\frac{\partial\epsilon_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial\epsilon_{12}}{\partial y} + \text{algebraic terms} = 0$$

$$\frac{1}{h_1}\frac{\partial\epsilon_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial\epsilon_{22}}{\partial y} + \text{algebraic terms} = 0$$

By using the assumed profile families, either singly or doubly infinite, we can express all quantities in these equations as functions of the two scale factors, δ and Δ, and the shape factors (two for singly infinite families; four for doubly infinite families). If we employ doubly infinite families, we assume herewith that appropriate compatibility conditions have been employed to eliminate two of the shape factors in favor of the other two shape factors and the two scale factors. We call the remaining shape factors χ and Σ. Then, in the case of greatest generality (with shared shape factors), we would know

$$\theta_{11} = \theta_{11}(\delta,\Delta,\chi,\Sigma), \qquad \theta_{12} = \theta_{12}(\delta,\Delta,\chi,\Sigma), \text{ etc.}$$

we would write

$$\frac{\partial\theta_{11}}{\partial x} = \frac{\partial\theta_{11}}{\partial\delta}\frac{\partial\delta}{\partial x} + \frac{\partial\theta_{11}}{\partial\Delta}\frac{\partial\Delta}{\partial x} + \frac{\partial\theta_{11}}{\partial\chi}\frac{\partial\chi}{\partial x} + \frac{\partial\theta_{11}}{\partial\Sigma}\frac{\partial\Sigma}{\partial x}$$

and similar expressions for the other derivatives. The resulting four equations for δ, Δ, χ, and Σ can be abbreviated by use of an index notation (summation over repeated indices) as

$$a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + C_i = 0, \qquad \begin{Bmatrix} i = 1, 2, 3, 4 \\ j = 1, 2, 3, 4 \end{Bmatrix} \tag{9}$$

We identify

$$u_1 = \delta, \qquad u_2 = \Delta, \qquad u_3 = \chi, \qquad u_4 = \Sigma$$

$$a_{ij} = \frac{1}{h_1}\begin{Vmatrix} \frac{\partial\theta_{11}}{\partial\delta} & \frac{\partial\theta_{11}}{\partial\Delta} & \frac{\partial\theta_{11}}{\partial\chi} & \frac{\partial\theta_{11}}{\partial\Sigma} \\ \frac{\partial\theta_{21}}{\partial\delta} & \frac{\partial\theta_{21}}{\partial\Delta} & \frac{\partial\theta_{21}}{\partial\chi} & \frac{\partial\theta_{21}}{\partial\Sigma} \\ \frac{\partial\varepsilon_{11}}{\partial\delta} & \frac{\partial\varepsilon_{11}}{\partial\Delta} & \frac{\partial\varepsilon_{11}}{\partial\chi} & \frac{\partial\varepsilon_{11}}{\partial\Sigma} \\ \frac{\partial\varepsilon_{21}}{\partial\delta} & \frac{\partial\varepsilon_{21}}{\partial\Delta} & \frac{\partial\varepsilon_{21}}{\partial\chi} & \frac{\partial\varepsilon_{21}}{\partial\Sigma} \end{Vmatrix}$$

and

$$b_{ij} = \frac{1}{h_2} \begin{Vmatrix} \frac{\partial\theta_{12}}{\partial\delta} & \frac{\partial\theta_{12}}{\partial\Delta} & \frac{\partial\theta_{12}}{\partial\chi} & \frac{\partial\theta_{12}}{\partial\Sigma} \\ \frac{\partial\theta_{22}}{\partial\delta} & \frac{\partial\theta_{22}}{\partial\Delta} & \frac{\partial\theta_{22}}{\partial\chi} & \frac{\partial\theta_{22}}{\partial\Sigma} \\ \frac{\partial\varepsilon_{12}}{\partial\delta} & \frac{\partial\varepsilon_{12}}{\partial\Delta} & \frac{\partial\varepsilon_{12}}{\partial\chi} & \frac{\partial\varepsilon_{12}}{\partial\Sigma} \\ \frac{\partial\varepsilon_{22}}{\partial\delta} & \frac{\partial\varepsilon_{22}}{\partial\Delta} & \frac{\partial\varepsilon_{22}}{\partial\chi} & \frac{\partial\varepsilon_{22}}{\partial\Sigma} \end{Vmatrix}$$

Difficulties of Further Discussion

One can easily carry on formally, writing expressions for the characteristic directions of the set Eq. (9) to show that there are either 0, 2, or 4 real characteristics at each point, and to indicate plausibly that in cases of weak cross flow all real characteristic slopes are close to that of the inviscid streamline. The essential arbitrariness of the momentum-integral methods precludes more precise statements for the case of general cross-flow magnitude. This makes the numerical values of the coefficients a_{ij} and b_{ij} a result not only of physical necessity but also of personal choice (exercised in the parametrization of velocity profiles). Furthermore, if conditions exist in which some of the characteristics are imaginary, the criterion for such conditions does not seem to be expressible in terms of a simple physical concept (such as that of subsonic versus supersonic flow in gas dynamics). A feeling for the behavior of the characteristic curves appears to come only from extensive experience with actual calculations, and this experience does not yet exist.

Difficulties of Application of the Method of Characteristics

While in principle integration along characteristics (when these are real) is the preferred method of solution of equation sets such as Eq. (9), numerical execution is awkward (leading to either poor accuracy or an excessive amount of computation) when the characteristic

lines all have nearly the same slope, as in the weak cross-flow case. Such cases beg for a different scheme of approximation. Mager (1957) amplifies these points somewhat.

Furthermore, although they are described in principle by Courant and Friedrichs (1948), numerical procedures become very tedious when as many as four real characteristics exist, as may be the case for set (9).

XI. EFFECTS OF COMPRESSIBILITY

In keeping with our announced intention to concern ourselves primarily with geometrical rather than thermodynamical complications of boundary-layer theory, the following comments will be very brief. A more comprehensive survey of the formal aspects can be found in Mager's review article.

Compressibility can give rise to, or modify, secondary flows in three essential ways. One is seen in the onset of differential "buoyancy" accelerations, which arise in a variable-density boundary layer under the action of velocity-independent force fields (such as centrifugal or gravity fields). A sample problem would be to determine the boundary layer on a strongly heated body that is moving slowly in a horizontal direction. A survey of laminar flows with body forces is given by Ostrach (1964) in the same volume which contains Mager's survey. A second is easily visualized in terms of the simple quasi-inviscid force-balance explanation of streamline curvature in a three-dimensional boundary layer. Since it is ρu^2 times the geodesic curvature which (nearly) balances the transverse pressure gradient, the curvature of boundary-layer streamlines will be increased by processes such as viscous dissipation, which reduce ρ, and decreased by processes such as extreme wall cooling, which increase ρ. In an article on the applicability of Stewartson's transformation to three-dimensional boundary layers, J. C. Cooke (1961) deduced the rule of thumb that if (a) Pr = 1, (b) $\mu \propto T$, (c) cross flows are almost negligible, (d) the wall is insulated, and (e) the external flow Mach number is moderate, then the compressible-boundary-layer flow corresponds to an incompressible flow over the same surface, but with all pressure gradients enhanced by the factor $1 + (\gamma - 1)M^2/2$. On the other hand, for the boundary layers on highly cooled walls of yawed blunted cones in hypersonic flow, Vaglio-Laurin (1959) has shown that a weak-cross-flow assumption, which decouples the streamwise-momentum equation from the cross-flow momentum equation and allows the former to be reduced to two-dimensional form by Mengler's transformation, is often justified by the near-constancy of ρu^2 across the boundary layer. The third and

least-studied way in which compressibility may engender secondary flows involves the displacement effect of the boundary layer. While it has been possible to incorporate this effect fairly successfully into the theory of two-dimensional hypersonic boundary layers, the only literature dealing with three-dimensional cases (other than axisymmetric cases) appear to refer to corner-flow, or boundary-region, problems.

While we cannot elaborate here on the subject of displacement effects, we shall state the definition of displacement thickness of a three-dimensional steady boundary layer, following Lighthill (1958). We refer to Fig. 6, which attempts to show two surface streamlines of the inviscid flow, denoted by η and $\eta + d\eta$ in intrinsic coordinates, and originating at a nodal point of attachment, 0. Over each of these streamlines we sketch a "wall of surface normals," the two walls meeting over 0. At the point (ξ,η) we erect a rectangle with base $h_2\, d\eta$ along the ξ = constant curve, and height δ_* along the local normal (ζ-axis).

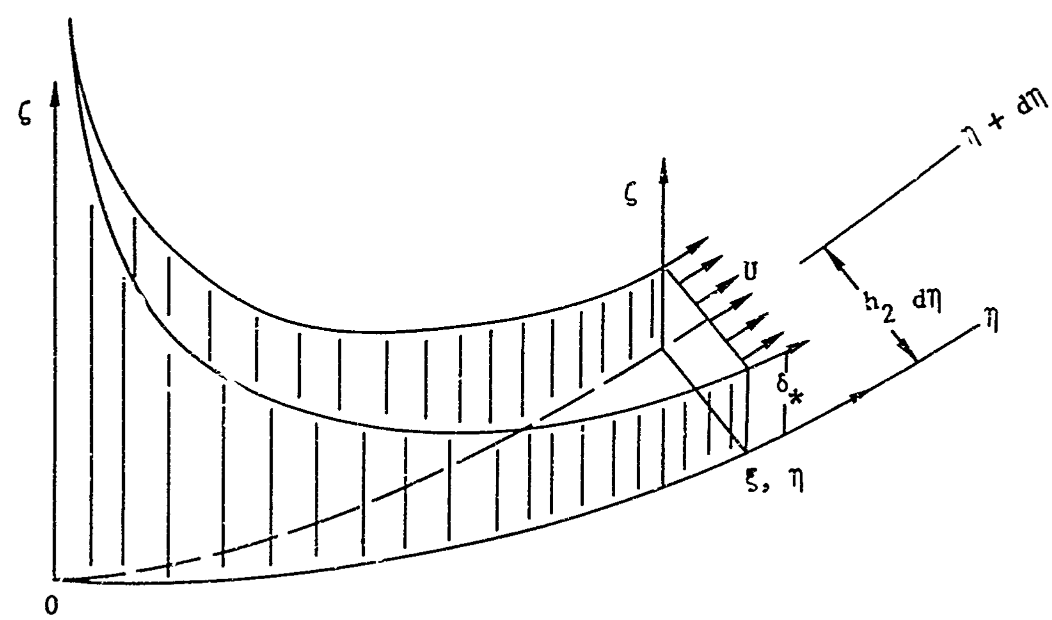


Fig. 6 -- Control volume which defines displacement thickness.

We may define the displacement thickness, δ_*, at ξ, η by equating the inviscid mass flow through the rectangle, $\rho_e U h_2 \, d\eta \, \delta_*$, to the mass flow "displaced by boundary-layer action" between the attachment node and ξ, η. Part of this displaced flow is accounted for by the deficit of streamwise flux in the boundary layer at ξ, η. This deficit, reckoned as a positive number, is $h_2 \, d\eta \int_o^\infty (\rho_e U - \rho u) \, d\zeta$. The remainder of the displaced flow has leaked out through "sidewalls" of normals, between 0 and ξ, due to boundary-layer cross flow. Between ξ' and $\xi' + d\xi'$ this deficit, again reckoned as a positive quantity, is $d\eta \, \partial/\partial\eta \, [h_1 \, d\xi' \int_o^\infty \rho v \, d\zeta]$, and the accumulated deficit between boundary-layer attachment and the point of interest is

$$d\eta \int_o^\xi \frac{\partial}{\partial\eta} \left[h_1(\xi',\eta) \int_o^\infty \rho(\xi') v(\xi') \, d\zeta \right] d\xi'$$

Thus we get

$$\rho_e U \delta_* h_2 \, d\eta = h_2 \, d\eta \int_o^\infty (\rho_e U - \rho u) \, d\zeta + d\eta \int_o^\xi \frac{\partial}{\partial\eta} \left[h_1 \int_o^\infty \rho v \, d\zeta \right] d\xi'$$

Introducing the "displacement integrals of the primary and secondary flows,"

$$\delta_1 \equiv \int_o^\infty \left(1 - \frac{\rho u}{\rho_e U}\right) d\zeta$$

and

$$\delta_2 \equiv \int_o^\infty \frac{\rho v}{\rho_e U} \, d\zeta$$

we get

$$\delta_* = \delta_1 + \frac{1}{\rho_e U h_2} \int_0^{\xi} \frac{\partial}{\partial \eta} \left[\rho_e U h_1 \delta_2 \right] d\xi'$$

In closing these brief remarks, we enter the observation that transformation methods, intended to carry compressible-flow problems over into previously solved incompressible-flow problems, appear to play a relatively minor role in the theory of three-dimensional boundary layers. Only in exceptional cases (e.g. Poots' (1965) analysis of the compressible stagnation point boundary layer) is an exact compressible-incompressible correlation established, while compressibility destroys the very feature (i.e., the independence principle) which underlies many of the available incompressible solutions.

XII. STABILITY OF THREE-DIMENSIONAL LAMINAR BOUNDARY LAYERS

The theory of hydrodynamic stability of three-dimensional boundary layers is reviewed by Stuart on pp. 549-558 of Chapter IX of L.B.L. An additional important survey is made by Brown in Vol. II of B.L.F.C.

The starting point of the usual theory is stated by Stuart (in Gregory, Stuart and Walker (1955)) as follows:

> "At a local station in the flow, the equations of stability, with certain approximations, were found to resemble formally those for two-dimensional flows; the relevant mean-flow velocity is the component in the direction of propagation of the disturbance at that particular station."*

If we assume that the spectrum of available infinitesimal disturbances is isotropic in direction, the stability theory for two-dimensional flows can be applied to the projections of the undisturbed velocity profile on various planes passing through the local wall normal, to find, for example, the minimum critical Reynolds number as a function of the angular orientation of the plane.

The most prominent result of such a study (examples are given by Brown and by Gregory, Stuart and Walker (1955)) is the destabilizing influence of secondary flow. In its presence, the velocity profile inevitably exhibits an inflexion point in many of its projections, as shown in Fig. 7.

Inflected profiles (see for example, Schlichting (1960) p. 387) have lower minimum critical Reynolds numbers and much larger areas of instability in the wave-number versus Reynolds-number plane than do profiles without inflexion. This fact partly determines the appropriate methods of analysis of the stability equation (see Brown (1961)). An infallible rule of thumb to indicate which projection of the velocity profile has the lowest critical Reynolds number does not seem to exist, but from examples shown by Brown it seems to lie close to the cross-flow ($\Phi = 90^{\circ}$) profile.

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*This assumption has recently been criticized by Lilly (1964) in a study of Eckman layer stability in a rotating tank.

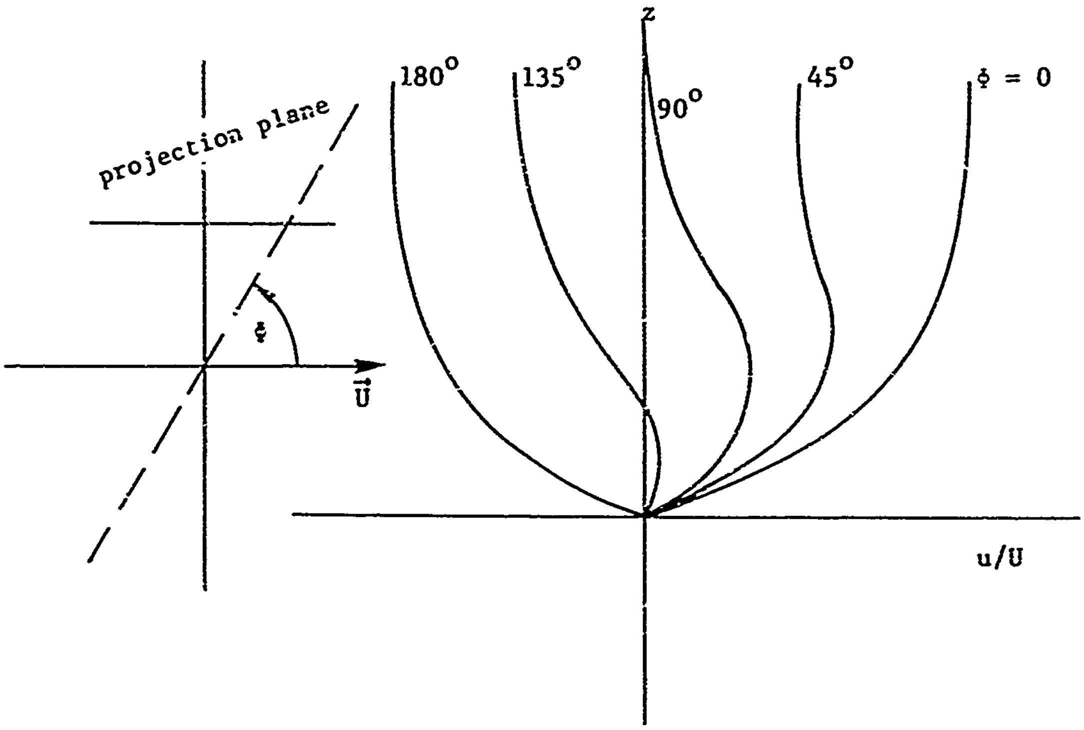


Fig. 7 -- Projections of the velocity profile on planes through the normal.

A projection of unique importance is that in which the inflexion point coincides with the crossover point (where the projected u is zero). Such a profile can support a stationary neutral disturbance (phase speed $C_r = 0$) whereas the usual neutral disturbances of two-dimensional boundary layers are traveling waves with $C_r > 0$. (See, for example, Fig. 16.12, p. 397 of Schlichting (1960)). Such stationary disturbances, having somehow appeared at particular locations on, say, a swept wing or a rotating disk (Gregory, Stuart and Walker (1955)), manifest their continued presence by a street-like pattern made visible by china-clay technique even substantially upstream of transition. Stuart's theory successfully predicts the "sweep angle" of these streets (which lie at right angles to the projection plane in which the inflexion and crossover points coincide), but gives poor estimates of the observed wavelengths. Eichelbrenner and Michel

(1958) have made observations of what may possibly be the same phenomenon occurring on a flattened ellipsoid. Finally, it has been speculated that even turbulent layers may be subject to this sort of instability, evidence for which is the regular streets of Sargassum or foam occasionally arrayed over large areas of the ocean surface (Faller, 1964) or by cloud rows in the atmosphere (Faller, 1965).

XIII. TRANSITION TO TURBULENCE

While it is known that transition to turbulence of the laminar boundary layer does not take place by simple amplification of two-dimensional disturbance waves, Smith (1956) was able to find a useful description in terms of stability theory parameters of a correlation of smooth-wall, low free-stream turbulence transition data made by Michel (1952). An attempt to extend the Michel-Smith transition criterion to make it applicable to three-dimensional boundary layers with weak cross flows has been described by Eichelbrenner and Michel (1958), and tested by them with china-clay observations of transition on a yawed ellipsoid of revolution. Because the comparison seemed less successful than one might have expected, particularly along the flanks of the ellipsoid, they then tried a second transition criterion, that of Owen and Randall (1952 and (1953), which appeared to be related to the instability of the secondary flow.

Owen and Randall's criterion is derived from experiments on yawed wings, and does not pretend to any great generaltiy. In its present state of development, it simply associates transition with a single critical value of a Reynolds number based on boundary-layer thickness and the maximum value of the cross-flow velocity:

$$Re_{tr} = v_{max} \frac{\delta}{\nu} \approx 200 - 300$$

where the range of values presumably comes from some as yet unexplained dependence on profile parameters.

When Eichelbrenner and Michel plotted points of $v_{max}\, \delta/\nu = 300$ on the surface of their ellipsoid of revolution, they found much better agreement with the observed transition curve at overall body Reynolds numbers $U_\infty L/\nu = 2$ to 6 million than they obtained with the Michel-Smith criterion. Of course, only the latter criterion could apply on lines of symmetry where $v_{max} = 0$, and in the vicinity of such lines it might be presumed to be the preemptive criterion.

Finally, Eichelbrenner and Michel remark that their observed transition lines frequently exhibit "tongues," the number of which appears

to increase with increasing Reynolds number. Similar tongues show in china-clay pictures of transition on swept wings, where they presumably develop from the stationary, swept laminar disturbances.

While the "tongues" of Eichelbrenner and Michel may be of a similar origin, the distances between them are not negligible compared to radii of curvature of the ellipsoid itself, so that one of the basic approximations of the standard "plane-flow" stability theory cannot be satisfied.

No detailed studies of the mechanics of transition of boundary layers with cross flow, comparable to those made by Schubauer, Klebanoff and others for the case of two-dimensional undisturbed flows, seem to exist.

XIV. THREE-DIMENSIONAL TURBULENT BOUNDARY LAYERS

On exterior aerodynamic surfaces, in rotating machinery, and in natural flows (rivers, the atmosphere, etc.) one has to deal with three-dimensional boundary layers that are turbulent. The theory of such boundary layers is only very slightly developed, and perhaps because of the formidable aspect of even the laminar-flow differential equations there has been almost no work along deductive lines, utilizing ad hoc models of eddy diffusivity or mixing length to provide the missing link between Reynolds stresses and mean-velocity derivatives. Exceptions can be found in the meteorological literature, but they typically involve special assumptions (negligibility of boundary-layer growth rate or of horizontal mean-flow convection which set them fairly far apart from the main-stream of boundary-layer theory.

Most studies of turbulent three-dimensional boundary layers have thus far taken an inductive and empirical approach, seeking out ways by which measured mean-velocity profiles can be represented parametrically, in hopes that momentum-integral methods can be devised to predict the evaluation of profile parameters for given bodies, surface conditions and external-flow velocities.

TIME-AVERAGED BOUNDARY-LAYER EQUATIONS

The boundary-layer approximations to the time-averaged Navier-Stokes equations, for the case of constant fluid density and viscosity, and for a steady but three-dimensional time-averaged flow, may be derived by the following sequence of assumptions.

The boundary layer is thin by comparison with any radius of curvature of the body surface, so that we may still assume $h_3 = 1$, $\partial h_1/\partial \zeta = \partial h_2/\partial \zeta = 0$ in the region of interest. With these simplifications the instantaneous momentum and continuity equations can be combined to give

$$\frac{\partial u}{\partial t} + \frac{1}{h_1}\frac{\partial}{\partial \xi}(u^2) + \frac{1}{h_2}\frac{\partial}{\partial \eta}(uv) + \frac{\partial}{\partial \zeta}(uw) + (u^2 - v^2)\varkappa_2 + 2uv\varkappa_1$$

$$+ \frac{1}{h_1}\frac{\partial}{\partial \xi}\left(\Lambda + \frac{p}{\rho} - \frac{1}{2}\omega^2 R^2\right) - 2\omega_3 v = \nu\left(\frac{\partial^2 u}{\partial \zeta^2} + \begin{array}{c}\text{other viscous terms which}\\ \text{are linear in the velocity}\\ \text{components}\end{array}\right)$$

$$\frac{\partial v}{\partial t} + \frac{1}{h_1}\frac{\partial}{\partial \xi}(vu) + \frac{1}{h_2}\frac{\partial}{\partial \eta}(v^2) + \frac{\partial}{\partial \zeta}(vw) + (v^2 - u^2)\varkappa_1 + 2vu\varkappa_2$$

$$+ \frac{1}{h_2}\frac{\partial}{\partial \eta}\left(\Lambda + \frac{p}{\rho} - \frac{1}{2}\omega^2 R^2\right) + 2\omega_3 u = \nu\left(\frac{\partial^2 v}{\partial \zeta^2} + \text{other linear viscous terms}\right)$$

and

$$\frac{\partial w}{\partial t} + \frac{1}{h_1}\frac{\partial}{\partial \xi}(wu) + \frac{1}{h_2}\frac{\partial}{\partial \eta}(wv) + \frac{\partial}{\partial \zeta}(w^2)$$

$$+ \frac{\partial}{\partial \zeta}\left(\Lambda + \frac{p}{\rho} - \frac{1}{2}\omega^2 R^2\right) + 2(\omega_1 v - \omega_2 u) = \nu\left(\frac{\partial^2 w}{\partial \zeta^2} + \begin{array}{c}\text{other linear}\\ \text{viscous terms}\end{array}\right)$$

In these equations

$$\varkappa_1 = \frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}, \qquad \text{and} \qquad \varkappa_2 = \frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}$$

are the geodesic curvatures of the surface coordinate curves.

Into these equations we now introduce the decomposition of turbulent variables into mean and fluctuating parts, and then average over time. For example, we write $u = \bar{u} + u'$, with the understanding that $\bar{\bar{u}} = \bar{u}$, $\bar{u}' = 0$. Then

$$\overline{u^2} = \overline{\bar{u}^2} + \overline{2\bar{u}u'} + \overline{u'^2} = \bar{u}^2 + \overline{u'^2}$$

We assume that $\partial\bar{u}/\partial t = 0$. To save space, we simply imagine this operation to be carried out.

We now assume that the thinness of the layer implies that if $1/h_1$ $\partial/\partial\xi(\bar{\ })$ and $1/h_2$ $\partial/\partial\eta(\bar{\ })$ are taken to be of order unity, then $\partial/\partial\zeta(\bar{\ })$ is an order of magnitude larger (say $O(L/\delta)$, where $\delta/L \ll 1$). The quantity in $(\bar{\ })$ may be any averaged property of the velocity field, like $\bar{u}$ or $(\overline{u'^2})$. For the moment we use this assumption only to establish the fact that the mean-flow viscous terms (e.g., $\nu\partial^2\bar{u}/\partial\zeta^2$), retain the same form as in a laminar boundary layer, and to establish, from the time-averaged continuity equation, that if $\bar{u}$ and $\bar{v}$ are taken to be of order unity, then $\bar{w}$ is of order $\delta/L \ll 1$. With these approximations the time-averaged boundary-layer equations become

$$\frac{1}{h_1}\frac{\partial}{\partial\xi}(\bar{u}^2) + \frac{1}{h_2}\frac{\partial}{\partial\eta}(\bar{u}\bar{v}) + \frac{\partial}{\partial\zeta}(\bar{u}\bar{w}) + (\bar{u}^2 - \bar{v}^2)\varkappa_2$$

$$+ 2\bar{u}\bar{v}\varkappa_1 + \frac{1}{h_1}\frac{\partial}{\partial\xi}\left(\Lambda + \frac{\bar{p}}{\rho} - \frac{1}{2}\omega^2R^2\right) - 2\omega_3\bar{v}$$

$$= \nu\frac{\partial^2\bar{u}}{\partial\zeta^2} - \left[\frac{1}{h_1}\frac{\partial}{\partial\xi}(\overline{u'^2}) + \frac{1}{h_2}\frac{\partial}{\partial\eta}(\overline{u'v'}) + \frac{\partial}{\partial\zeta}(\overline{u'w'})\right]$$

$$- (\overline{u'^2} - \overline{v'^2})\varkappa_2 - 2\overline{u'v'}\,\varkappa_1$$

$$\frac{1}{h_1}\frac{\partial}{\partial\xi}(\bar{v}\bar{u}) + \frac{1}{h_2}\frac{\partial}{\partial\eta}(\bar{v})^2 + \frac{\partial}{\partial\zeta}(\bar{v}\bar{w}) + (\bar{v}^2 - \bar{u}^2)\varkappa_1$$

$$+ 2\bar{v}\bar{u}\varkappa_2 + \frac{1}{h_2}\frac{\partial}{\partial\eta}\left(\Lambda + \frac{\bar{p}}{\rho} - \frac{1}{2}\omega^2R^2\right) + 2\omega_3\bar{u}$$

$$= \nu\frac{\partial^2\bar{v}}{\partial\zeta^2} - \left[\frac{1}{h_1}\frac{\partial}{\partial\xi}(\overline{v'u'}) + \frac{1}{h_2}\frac{\partial}{\partial\eta}(\overline{v'^2}) + \frac{\partial}{\partial\zeta}(\overline{v'w'})\right]$$

$$- (\overline{v'^2} - \overline{u'^2})\varkappa_1 - 2\overline{v'u'}\,\varkappa_2$$

and

$$\frac{\partial}{\partial\zeta}\left(\Lambda + \frac{\bar{p}}{\rho} - \frac{1}{2}\omega^2R^2\right) + 2(\omega_1\bar{v} - \omega_2\bar{u}) =$$

$$-\left[\frac{1}{h_1}\frac{\partial}{\partial\xi}\overline{(w'u')} + \frac{1}{h_2}\frac{\partial}{\partial\eta}\overline{(w'v')} + \frac{\partial}{\partial\zeta}\overline{(w'^2)}\right]$$

To make further simplifications we must have additional experimental knowledge, regarding the typical orders of magnitude of the kinematic Reynolds stresses (e.g., u'^2, $\overline{u'v'}$) and their derivatives. This information is not directly available for three-dimensional turbulent boundary layers, so we shall have to employ a liberal amount of guesswork. We should note with particular care that while v'^2 and u'^2 are probably roughly independent of our choice of surface coordinate system (in a two-dimensional boundary layer, streamwise fluctuation and cross-stream fluctuation are roughly equally energetic), $\bar{v}^2$ and $\bar{u}^2$ do depend on this choice. We shall proceed in intrinsic coordinates, and shall assume that all kinematic Reynolds stress components are of the same order of magnitude, which we estimate (from Klebanoff's data for a zero pressure gradient two-dimensional boundary layer) to be no more than 1 percent of the local U^2. Thus we shall drop $1/h_1\ \partial/\partial\xi\ (\overline{u'v'})$ and $1/h_2\ \partial/\partial\eta\ (v')^2$ compared to $\partial/\partial\zeta\ (\overline{v'w'})$ in the η-momentum equation, and make corresponding simplifications in the other equations. The ζ-momentum equation can be integrated over ζ to give

$$P(\xi,\eta,\zeta) \equiv \Lambda + \frac{\bar{p}}{\rho} - \frac{1}{2}\omega^2R^2 = P_o(\xi,\eta,\infty) - \overline{w'^2} + 2\int_o^\delta (\omega_2\bar{u} - \omega_1\bar{v})\, d\zeta$$

With the assumption that any coordinate spin is moderate, so that $\omega U\delta$ is at most of order $\delta/L \ll 1$, we can drop the integral. Then, when $\partial P_o/\partial\xi$ and $\partial P_o/\partial\eta$ are evaluated from the inviscid momentum equation, we shall be comparing $\partial/\partial\xi\ (U^2)$ or $U^2\kappa_1$ with $\partial/\partial\xi\ w'^2$, and we assume that the fluctuation term can be dropped in that comparison. Substituting

for $\partial P_o/\partial \xi$ and $\partial P_o/\partial \eta$ in the ξ- and η-momentum equations, and using the continuity equation again, we find the forms

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{u}}{\partial \xi} - \frac{U}{h_1}\frac{\partial U}{\partial \xi} + \frac{\bar{v}}{h_2}\frac{\partial \bar{u}}{\partial \zeta} + \bar{w}\frac{\partial \bar{u}}{\partial \zeta} - \bar{v}^2 \varkappa_2 + \bar{u}\bar{v}\varkappa_1 - 2\omega_3 \bar{v}$$

$$= \nu \frac{\partial^2 \bar{u}}{\partial \zeta^2} - \frac{\partial}{\partial \zeta}(\overline{u'w'}) - (\overline{u'^2} - \overline{v'^2})\varkappa_2 - 2\overline{u'v'}\varkappa_1 \tag{10}$$

and

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{v}}{\partial \xi} + \frac{\bar{v}}{h_2}\frac{\partial \bar{v}}{\partial \eta} + \bar{w}\frac{\partial \bar{u}}{\partial \zeta} + (U^2 - \bar{u}^2)\varkappa_1 + \bar{u}\bar{v}\varkappa_2 + 2\omega_3 \bar{u}$$

$$= \nu \frac{\partial^2 \bar{v}}{\partial \zeta^2} - \frac{\partial}{\partial \zeta}(\overline{v'w'}) - (\overline{v'^2} - \overline{u'^2})\varkappa_1 - 2\overline{u'v'}\varkappa_2 \tag{11}$$

We reserve judgment on the importance of the last two terms in each equation until we have integrated these equations over ζ.

MOMENTUM-INTEGRAL EQUATIONS

We note the boundary conditions at $\zeta = 0$,

$$\bar{u} = \bar{v} = 0 \qquad \text{(no slip)}$$

$$\bar{w} = \bar{w}_s \qquad \text{(given blowing velocity)}$$

$$\overline{u'w'} = \overline{v'w'} = 0 \qquad \text{(for a smooth wall)}$$

$$\nu \frac{\partial \bar{u}}{\partial \zeta} - \overline{u'w'} = \frac{U^2}{2} c_{f_1}$$

and

$$\nu \frac{\partial \bar{v}}{\partial \zeta} - \overline{v'w'} = \frac{U^2}{2} C_{f_2} \qquad \text{(at a rough wall)}$$

At $\zeta = \infty$,

$$\bar{u} = U, \qquad \bar{v} = 0$$

$$\frac{\partial \bar{u}}{\partial \zeta} = \frac{\partial \bar{v}}{\partial \zeta} = \overline{u'w'} = \overline{v'w'} = 0$$

We define the integral thickness as

$$L\delta_1 \equiv \int_o^\infty \left(1 - \frac{\bar{u}}{U}\right) d\zeta, \qquad L\delta_2 \equiv - \int_o^\infty \frac{\bar{v}}{U} d\zeta$$

$$L\theta_{11} \equiv \int_o^\infty \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}}{U}\right) d\zeta, \qquad L\theta_{22} \equiv - \int_o^\infty \frac{\bar{v}^2}{U^2} d\zeta$$

$$L\theta_{12} \equiv \int_o^\infty \frac{\bar{v}}{U}\left(1 - \frac{\bar{u}}{U}\right) d\zeta, \qquad L\theta_{21} \equiv - \int_o^\infty \frac{\bar{v}\bar{u}}{U^2} d\zeta = L\theta_{12} + L\delta_2$$

$$LF_{11} \equiv \int_o^\infty \left(\frac{\overline{u'^2} - \overline{v'^2}}{U^2}\right) d\zeta, \qquad LF_{12} \equiv \int_o^\infty \left(\frac{\overline{u'v'}}{U^2}\right) d\zeta$$

Next we add $(\bar{u} - U)$ times the time-averaged continuity equation to (10) and $\bar{v}$ times the continuity equation to Eq. (11). This gives

$$\left(\frac{\bar{u} - U}{h_1}\right)\frac{\partial U}{\partial \xi} + \frac{1}{h_1}\frac{\partial}{\partial \xi}[\bar{u}(\bar{u} - U)] + \frac{1}{h_2}\frac{\partial}{\partial \eta}[\bar{v}(\bar{u} - U)] + \frac{\bar{v}}{h_2}\frac{\partial U}{\partial \eta} + \frac{\partial}{\partial \zeta}[\bar{w}(\bar{u} - U)]$$

$$+ [(\bar{u} - U)\bar{u} - \bar{v}^2]\kappa_2 + [(\bar{u} - U)\bar{v} + \bar{u}\bar{v}]\kappa_1 - 2\omega_3\bar{v} = \nu \frac{\partial^2 \bar{u}}{\partial \zeta^2}$$

$$- \frac{\partial}{\partial \zeta}(\overline{u'w'}) - (\overline{u'^2} - \overline{v'^2})\kappa_2 - 2\overline{u'v'}\kappa_1$$

and

$$\frac{1}{h_1}\frac{\partial}{\partial\zeta}(\bar{u}\bar{v}) + \frac{1}{h_2}\frac{\partial}{\partial\eta}(\bar{v})^2 + \frac{\partial}{\partial\zeta}(\bar{w}\bar{v}) - (U^2 - \bar{u}^2 + \bar{v}^2)\varkappa_1 + 2\bar{u}\bar{v}\varkappa_2$$

$$+ 2\omega_3(\bar{u} - U) = \nu\frac{\partial^2\bar{v}}{\partial\zeta^2} - \frac{\partial}{\partial\zeta}(\overline{v'w'}) - (\overline{v'^2} - \overline{u'^2})\varkappa_1 - 2\overline{u'v'}\varkappa_2$$

All terms will now give finite values when integrated from $\zeta = 0$ to $\zeta = \infty$. (In these integrations U and h_1, h_2 are taken independent of ζ.) After division through by U^2 and introduction of the symbols

$$x = \frac{\xi}{L}, \qquad y = \frac{\eta}{L}$$

(and other symbols such as m, n, λ, $\varkappa$, as used above), we obtain

$$\frac{1}{h_1}\frac{\partial\theta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{12}}{\partial y} = \frac{C_{f_1}}{2} - \frac{1}{h_1 x}\left[m(2\theta_{11} + \delta_1) + \varkappa(\theta_{11} - \theta_{22} - F_{11})\right]$$

$$- \frac{2\lambda}{h_2 y}F_{12} + \frac{2\Omega_3 L}{U}\theta_{12} + \left(\frac{2\omega_3 + \Omega_3}{U}\right)L\delta_2 + \frac{\bar{w}_g}{U}$$

and

$$\frac{1}{h_1}\frac{\partial\theta_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial\theta_{22}}{\partial y} = \frac{C_{f_2}}{2} - \frac{2}{h_1 x}\left[(m + \varkappa)\theta_{21} - \varkappa F_{12}\right]$$

$$- \frac{1}{h_2 y}\left[\lambda(\theta_{22} + \theta_{11} + F_{11}) + n\delta_1\right]$$

$$+ \frac{2\Omega_3 L}{U}\theta_{22} - \left(\frac{2\omega_3 + \Omega_3}{U}\right)L\delta_1$$

Note: Except for the presence of the blowing velocity and the "fluctuation integrals" F_{11} and F_{12}, these equations agree exactly with the laminar flow Eqs. (5) and (6). The apparent differences in their respective right-hand numbers came from the fact that in our laminar-flow treatment we have measured δ's and θ's in units of z, where $z = \sqrt{UL/\nu x}(\zeta/L)$, while in the present treatment they are measured simply in units of ζ/L. This means that

$$\frac{\partial}{\partial x}\theta_{11,21}(z) = \frac{\partial}{\partial x}\theta_{11,21}\left(\frac{\zeta}{L}\right) + \frac{n-1}{2x}\theta_{11,21}\left(\frac{\zeta}{L}\right)$$

$$\frac{\partial}{\partial y}\theta_{12,22}(z) = \frac{\partial}{\partial y}\theta_{12,22}\left(\frac{\zeta}{L}\right) + \frac{n}{2y}\theta_{12,22}\left(\frac{\zeta}{L}\right)$$

Similarly we can show that

$$\frac{f''(z=0)}{\theta_{11}(z)} = \frac{x}{2}\frac{C_{f_1}}{\theta_{11}\left(\frac{\zeta}{L}\right)}$$

In their present form (with δ's and θ's measured in units of ζ/L) and with the suction-velocity term $\tilde{w}_s/U$, the equations are easily specialized back to their familiar forms for two-dimensional boundary layers ($h_1 = 1$, $\varkappa = \lambda = \Omega_3 = \omega_3 \approx 0$). Then the streamwise momentum-integral equation becomes

$$\frac{d\theta_{11}}{dx} = \frac{C_{f_1}}{2} - (2\theta_{11} + \delta_1)\frac{m}{x} + \frac{\tilde{w}_s}{U}$$

ELIMINATION OF THE FLUCTUATION INTEGRALS

If we can make order-of-magnitude estimates from data on two-dimensional boundary layers, it appears as though F_{11} may not exceed 2 percent of θ_{11}, and hence it may be dropped wherever the two quantities appear as a sum or difference. (This relies on $\theta_{11} + \theta_{22}$ being not much smaller than θ_{11}, an assumption which could only be violated in case of extraordinarily strong cross flows.)

We cannot so decisively dispose of F_{12} by comparison with θ_{12} or θ_{21}, since it is hard to decide on typical orders of magnitude for either quantity. Since further development of the theory would be nearly impossible in a case in which F_{12} was important, we shall arbitrarily ignore it. The equations which we shall discuss further are then cast entirely in terms of integral quantities which can be derived from assumed profiles of $\bar{u}$ and $\bar{v}$ and the friction coefficients C_{f_1} and C_{f_2}.

COUNT OF UNKNOWNS AND EQUATIONS

If we accept the approximations whereby "fluctuation integrals" such as F_{11} and F_{12} are neglected, we are left with two differential equations for the determination of seven unknowns (five independent integral thicknesses and two skin friction coefficients).

For subtle reasons which could hardly be anticipated before an examination of experimental u and v profiles (see Rotta (1962) pp. 70-73 and pp. 172-181), the compatibility conditions at the wall do not provide useful additional relations between the δ's, θ's, and say, the given pressure gradient.

Additional weighted momentum-integral equations can of course be formed, but we can see at a glance that these will involve important integral thicknesses in which the Reynolds stresses appear in the integrands. For example, the u-weighted ξ-momentum equation will involve the thickness

$$\int_0^\infty \frac{\partial \bar{u}}{\partial \zeta} (\overline{u'w'})\, d\zeta$$

which describes part of the conversion of kinetic energy from the "mean-flow budget" to the "fluctuation budget." Since the Reynolds stresses are harder to measure, with acceptable accuracy, than the $\bar{u}$ and $\bar{v}$ profiles from which the δ's and θ's are computed, there does not yet exist an extensive empirical base for work with the weighted integral equations. If we abandon them, we are left to devise five additional relations between the δ's, θ's, C_f's and our given parameters, by direct examination of experimental data.

EMPIRICAL GENERALIZATIONS ABOUT TWO-DIMENSIONAL TURBULENT BOUNDARY-LAYER PROFILES

In many cases, especially those with weak cross flows, we expect the streamwise velocity profile to be quite like those which have been so extensively studied in flows possessing a carefully cultured two-dimensionality of the time-averaged state. The properties of such flows have been reviewed in detail recently by Rotta (1962) and Schubauer and Tchen (1959). We are mostly interested in the empirical generalizations for $\bar{u}$ which are reviewed by Rotta on pp. 156-166. In particular, we select for further discussion the doubly-infinite family of $\bar{u}$ profiles given by the wall-wake model of Coles (1956).

The Wall-Wake Model of $\bar{u}$ Profiles

According to this model, which though frankly empirical and approximate, is capable of fitting experimental $\bar{u}$ profiles with remarkable accuracy throughout a wide range of both favorable and adverse pressure gradients, the profile is constructed by a linear combination of two universal functions. One of these, called the wall function or the law of the wall, relates u only to the local friction velocity, $u_\tau \equiv \sqrt{\tau_w/\rho}$, the kinematic viscosity ν, and the normal distance ζ, provided that the wall is smooth. For nonzero skin friction, this law dominates the $\bar{u}$ profile over some finite range of ζ, and within this region the local pressure gradient has no direct effect (within the broad range of conditions on which the law is based). By dimensional analysis, this law can be written

$$\frac{\bar{u}}{u_\tau} = f\left(u_\tau \frac{\zeta}{\nu}\right)$$

When the wall is rough, and if the roughness elements possess geometrical similarity in their shape and spacing so that they can be characterized by a roughness height k_r, the law may be generalized to

$$\frac{\bar{u}}{u_\tau} = f_1\left(u_\tau \frac{\zeta}{\nu}, \quad u_\tau \frac{k_r}{\nu}\right)$$

In some cases, particularly those involving favorable pressure gradients or large Reynolds numbers, the wall function may suffice to describe $\bar{u}$ over a sufficient range of $u_\tau\ \zeta/\nu$ so that, for $u_\tau\ \zeta/\nu \gtrsim 50$, it assumes the famous logarithmic form

$$f\left(\frac{u_\tau \zeta}{\nu} > 50\right) = \frac{1}{K}\,\ell n\left(\frac{u_\tau \zeta}{\nu}\right) + C$$

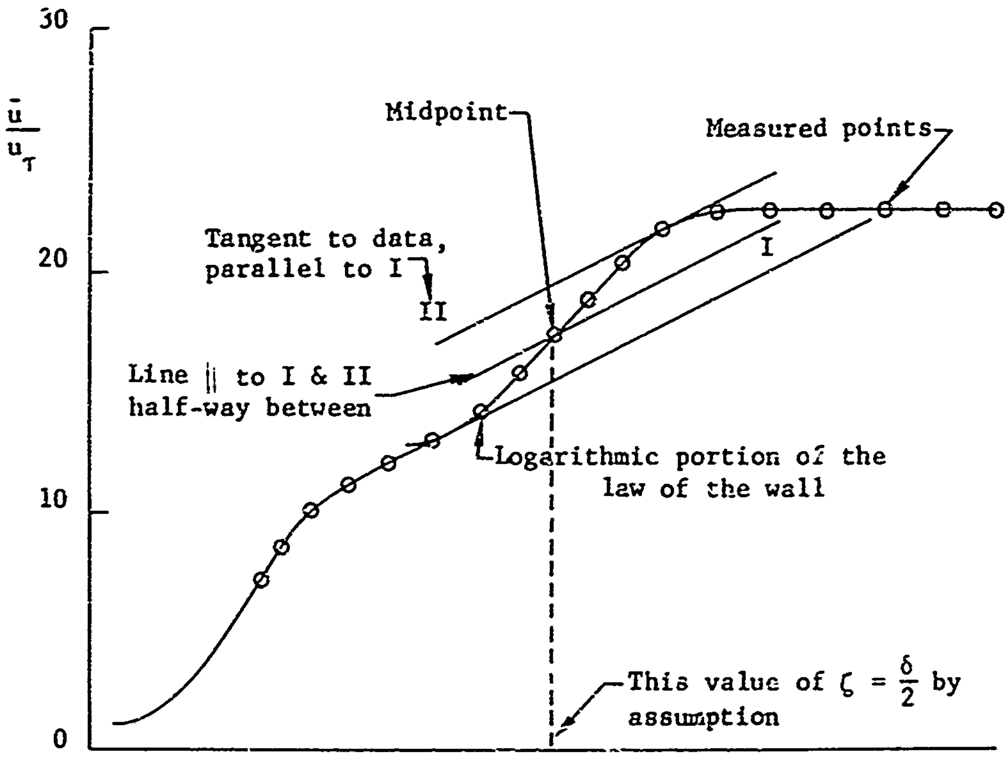
with $K = 0.41$ and, for smooth walls, $C = 5.0$, as found from experiment. When this is attained for a rough wall, the effect of roughness is entirely in the "constant" C:

$$f_1 = \frac{1}{K}\,\ell n\left(\frac{u_\tau \zeta}{\nu}\right) + C\left(\frac{u_\tau k_r}{\nu}, \text{ roughness shape}\right) \qquad \left(u_\tau \frac{\zeta}{\nu} > 50,\ \zeta > k_r\right)$$

The dependence of C on $u_\tau k_r/\nu$ is independent of shape, but for Nikuradse's sand grains, $u_\tau k_r/\nu \lesssim 5$ gave $C = 5.0$, or smooth flow, while for $u_\tau k_r/\nu \gtrsim 70$, fully rough flow with $C = C_r$ (shape only) $(1/K)\ \ell n\ (u_\tau k_r/\nu)$, so that $\bar{u}/u_\tau = (1/K)\ \ell n\ (\zeta/k_r) + C_r$ (fully rough).

The direct influence of local pressure gradient and boundary-layer thickness enter $\bar{u}$ through the second, or wake, component of the profile. The form of this was discovered by Coles (1956) by a procedure indicated in Fig. 8. Only profiles for which the logarithmic portion of the wall law was empirically well established could be used initially. The wake component was determined by subtracting the extrapolated (logarithmic) wall law from the measured $\bar{u}$, and a definite procedure for defining the boundary layer thickness δ was established (as seen in Fig. 8). Ideally u_τ as well as $\bar{u}$ should be directly measured in the experiment.

The experimental residual, $\bar{u} - u_\tau f$, was discovered by Coles to have a universal shape in boundary layers under a free stream of sufficiently low (< 0.1 percent) turbulence level. With δ defined by



ζ (actual measured distance from wall, plotted on a logarithmic scale)

Fig. 8 -- Law of the wake.

Fig. 8, and with a universal wake function $W(\zeta/\delta)$ normalized to run from 0 to 2 as ζ goes from 0 to δ, Coles' complete wall-wake model for $\bar{u}$ is given by

$$\frac{\bar{u}}{u_\tau} = f_1\left(\frac{u_\tau \zeta}{\nu}, \quad \frac{u_\tau k_r}{\nu}\right) + \frac{\pi}{K} W\left(\frac{\zeta}{\delta}\right), \qquad 0 \le y \le \delta$$

In this the dimensionless parameter $\pi(\xi)$ assumes values determined by the local pressure gradient and prior boundary-layer history, and K is again the von Karman constant. We can re-express it in terms of the given U (instead of the unknown u_τ) as

$$\frac{\bar{u}}{U} = \sqrt{\frac{C_f}{2}}\left\{f_1\left[\sqrt{\frac{C_f}{2}}\left(\frac{UL}{\nu}\right)\frac{\zeta}{L}, \quad \sqrt{\frac{C_f}{2}}\left(\frac{UL}{\nu}\right)\frac{k_r}{L}\right] + \frac{\pi}{K} W\left(\frac{L}{\delta}\frac{\zeta}{L}\right)\right\}$$

We see that this forms a doubly-infinite family of profile shapes, governed by the three dependent-variable parameters δ/L, C_f, and π and the two given parameters UL/ν and k_r/L. By setting $\zeta = \delta$ we get one relation between these parameters, namely

$$\boxed{\sqrt{\frac{2}{C_f}} = f_1\left[\sqrt{\frac{C_f}{2}}\left(\frac{UL}{\nu}\right)\frac{\delta}{L}, \quad \sqrt{\frac{C_f}{2}}\left(\frac{UL}{\nu}\right)\frac{k_r}{L}\right] + \frac{2\pi}{K}}$$

This difficult implicit relationship is untangled graphically by Rotta (1962) on pp. 168 and 169 (for C = 5.2).

The integrals for δ_1 and θ_{11} can now be carried out by noting that $\int_o^1 W(\phi)\, d\phi = 1$, and ignoring deviations of f_1 from its asymptotic logarithmic form, so that $\int_o^1 \zeta\, \partial f_1/\partial\zeta\, d(\zeta/\delta) \approx 1$. We get

$$\delta_1 = \delta\sqrt{\frac{C_f}{2}}\left(\frac{1+\pi}{K}\right)$$

and

$$\theta_{11} = \delta\left\{\sqrt{\frac{C_f}{2}}\left(\frac{1+\pi}{K}\right) - \frac{C_f}{K^2}\left(1 + \alpha\pi + \beta\pi^2\right)\right\}$$

where, according to Coles, $\alpha = 1.600$ and $\beta = 0.761$.

The "Missing Relation"

When δ_1 and θ_{11} are eliminated in favor of δ, C_f, and π, the momentum-integral equation plus the free-stream relation (boxed above) give us two equations for three unknowns.

Neither of these equations postulates any direct local connection between pressure gradient and profile shape, and we might hope to find something like this empirically, perhaps in the form of a correlation between π and $\delta_1(\partial p/\partial\xi)/\tau_o$. Each of these quantities is constant (Coles 1956, Clauser 1956) in so-called equilibrium or self-preserving boundary layers, so that a few points on a correlation curve are readily

available. Of course, it is naive to hope that a local correlation of this type might hold very generally in boundary layers in which π changes radically with ξ, but even a rough correlation would probably lead to results as good as those obtainable with other popular methods--e.g., that of von Doenhoff and Tetervin (see Schubauer and Tchen, 1959). Good accuracy would be expected in favorable pressure gradients (where π is small, ranging between zero in converging channel flows to 0.55 with zero pressure gradient), and poorer accuracy would be expected as dp/dx becomes increasingly positive and separation is approached. In the latter case the detailed history of boundary-layer growth is bound to play an increasingly important role, even more so than in laminar layers. In the extreme, Clauser (1956) discovered that the profile in strong adverse pressure gradients was so sensitive to upstream history that it could not be effectively manipulated by changing the local $\partial p/\partial \xi$. This phenomenon is occasionally referred to as instability, in the sense that a small local change in $\partial p/\partial \xi$ (or in wall roughness) may produce a downstream effect which grows, rather than diminishes, with increasing downstream distance.

Finally, we note with Rotta (1962), p. 199, that the "missing relation," as we have called it, must be an independent empirical discovery, and that it cannot be obtained by any manipulation of the equations (momentum and continuity) and profile information ($\bar{u}(\zeta/\delta$, δ/L, C_f, π)) we already possess. For example, even though Coles has shown that the wall-wake model can be applied to the momentum equation, and accurate values of Reynolds stresses computed, the resulting stress profile cannot be inserted into (say) the energy integral equation to obtain our missing relation. What results from such a procedure is an identity (1 = 1) and not an independent equation.

EMPIRICAL MODELS FOR THREE-DIMENSIONAL TURBULENT BOUNDARY-LAYER PROFILES

We found in our review of the two-dimensional turbulent boundary layer that there exists a profile model, the wall-wake model, which accurately fits a very wide variety of profiles by use of two universal functions and three parameters, even though we do not yet possess

an effective way of predicting these parameters. We now wish to see whether in the three-dimensional case we can go equally far, discovering widely applicable models for both the streamwise and cross-flow velocity profiles. As might be expected from the comparatively small amount of study yet given to three-dimensional boundary layers, the answer seems to be "not yet." Two major proposals, both of which seem to have some experimental support, have been made; they agree in some cases and disagree in others.

Generalization of the Wall-Wake Model

In his 1956 article Coles suggested a straightforward vector generalization of the wall-wake model into

$$\vec{u} = \vec{u}_\tau f_1 \left(\frac{u_\tau \zeta}{\nu}, \quad \frac{u_\tau k_r}{\nu}\right) + \underline{\underline{\pi}}\, \frac{\vec{u}_\tau}{K} W \left(\frac{\zeta}{\delta}\right)$$

where f_1 and W are the same universal functions as in the two-dimensional case, and $\underline{\underline{\pi}}\ (\zeta,\eta)$ has become a tensor parameter, which operates on the friction velocity to beget another vector tangential to the wall. The friction velocity vector has the direction of the skin friction vector, and its magnitude is, as before, $u_\tau = \sqrt{\tau_o/\rho}$.

Experimental testing of this model requires accurate measurements of both magnitude and direction of velocity and skin friction, and it appears that instrumentation employed so far is somewhat cruder than that used in two-dimensional flow studies. Be that as it may, the test would in principle consist of two steps: (a) a check of the parallelism of the wake-component vectors, as visualized in the polar plot of Fig. 9; (b) a plot of the wake-component magnitude versus ζ. If the data pass the first test (parallelism of wake-component vectors) we then see whether the amplitude of the wake component can be described by a universal profile function. Finally, if this wake profile shows the symmetry of $W(\zeta/\delta)$, δ is easily determined, and so is the amplitude of the vector $\underline{\underline{\pi}}\, \vec{u}_\tau$.

Since 1956, various experimenters have compared data with the generalized wall-wake model; while none of their comparisons quite live up

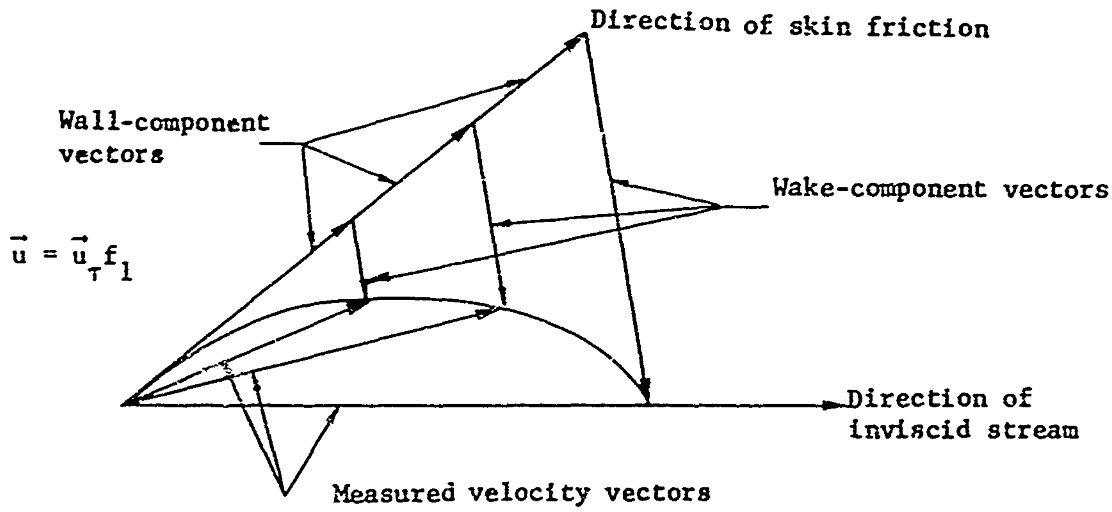


Fig. 9 -- Projection of velocity profile on tangent plane.

to the conditions we have outlined (in particular, $\vec{u}_\tau$ is almost never directly measured) they appear to provide fair confirmation of Coles' ideas about the wall component, and fair to poor confirmation of the proposed behavior of the wake component. (See Perry and Joubert (1965), Hornung and Joubert (1963), and Johnston (1960).)

Johnston's Triangle Model

Taylor and his student Johnston (1960), presented in 1959 and 1960, respectively, a striking empirical generalization about the polar plots of three-dimensional turbulent boundary-layer profiles. This has become known as the triangle law, for reasons seen in Fig. 10.

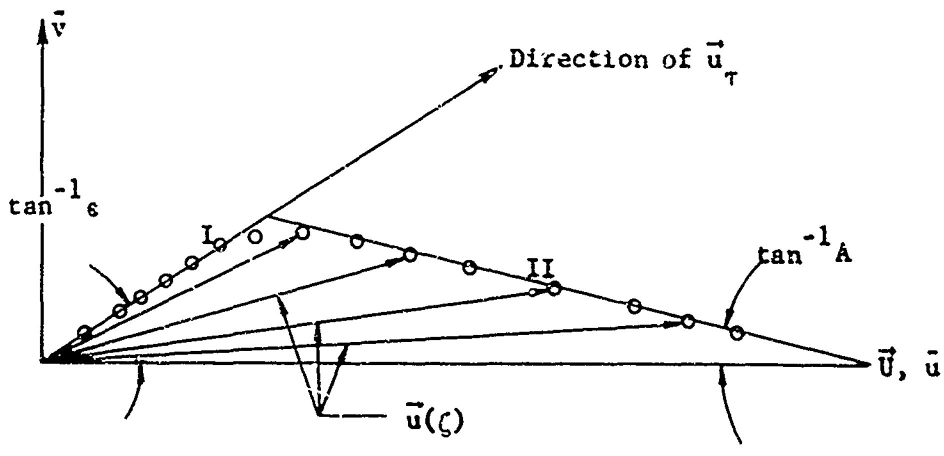


Fig. 10 -- The "Triangle Law."

Many polar plots were found to be nearly triangles, with base given by the inviscid velocity vector $\vec{U}$, one side aligned along the skin friction direction, and the third making an angle with $\vec{U}$ which appeared in many cases to be simply related to the total angle through which the inviscid streamlines had turned after a point in which the flow had originally been two-dimensional. The triangle law is thus a simple statement of the dependence of $\bar{v}$ on $\bar{u}$, without direct reference to ζ. It is a two-part statement. In Part I, which is clearly related to Coles' wall-dominated region,

$$\bar{v} = \epsilon\bar{u}$$

In Part II, the outer region,

$$\bar{v} = A(U - \bar{u})$$

The corner at which I and II join, was estimated originally by Johnston to fall at $u_\tau\zeta/\nu \sim 16$, but much higher estimates (~ 150) are given by Hornung and Joubert (1963).

Conflict Between the Two Models

While certain sets of data (for example, that of Kuethe et al. (1949), which Coles first examined) seem to fit both models fairly well, it can readily be seen that the two models are in fairly direct conflict. For example, we can show that if Coles' wake-component vectors are indeed parallel, and if Johnston's triangle rule is rigorously applied, the wake-component magnitudes cannot be described by an S-shaped function like $w(\xi/\delta)$. Under the assumed conditions, the angles α and β in the polar plot (Fig. 11) are independent of ζ. Thus we get

$$u_{wake}(\zeta) = \frac{\sin\alpha}{\sin\beta}[u_{wall}(\zeta) - u_{wall}(\zeta_1)]$$

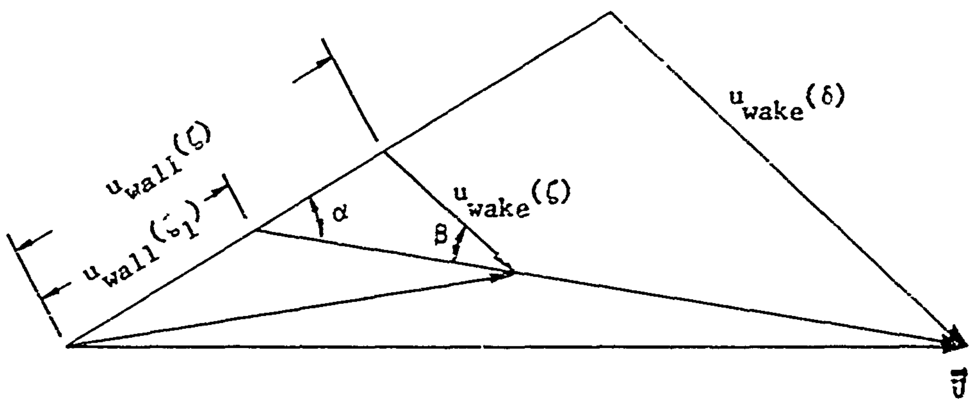


Fig. 11 -- Hypothetical projected velocity profile.

If we assume that the wall law is logarithmic for $\zeta > \zeta_1$ and define $W = 2u_{wake}(\zeta)/u_{wake}(\delta)$, we find that

$$W = 2\,\ell n\left(\frac{\zeta}{\zeta_1}\right)\Big/\ell n\left(\frac{\delta}{\zeta_1}\right)$$

Curiously, this gives a fair fit to Hornung and Joubert's wake function data, as shown in their Fig. 7, with the reasonable choices of $\delta/\zeta_1 = 20$ or 30! Coles has suggested that the generalized wall-wake model is most likely to succeed in cases in which the divergence of Reynolds stress is reasonably large compared with the lateral component of inertial force, i.e., when $\partial\tau/\partial\zeta \gtrsim \rho u^2 \kappa_1$, and he estimates $\rho u^2 \kappa_1/(\partial\tau/\partial\zeta)$ to be about 250 $\kappa_1 \delta$. If the radius of curvature of surface streamlines is much shorter than a few hundred boundary-layer thicknesses, the outer part of the boundary layer accelerates as a more or less inviscid, although rotational, flow. The latter situation has been assumed by various authors to be the one for which the triangle model will be most successful.

SUMMARY

One could go on to point out sample flows for which Johnston's triangle rule cannot conceivably apply, but perhaps it is clear from what has been already said that the prognosis for successful computa-

tions of three-dimensional turbulent boundary layers is quite poor at present. There are undoubtedly problems involving favorable pressure gradients and gentle streamline curvatures in which success nearly comparable to that available in two-dimensional problems is possible. For example, Eichelbrenner's (1963) extensive computations for ring wings, and those of Vaglio-Laurin (1959) for reentry bodies, use very simple profiles and skin-friction laws. Turbulence complicates the problem in a fundamental way, but it also probably suppresses anomalous cross flows by its vigorous momentum exchanges, and qualitatively can be expected to have a beneficial effect upon the prevention or delay of separation.

It seems that detailed experimental work on three-dimensional turbulent boundary layers is not only needed in its own right, but may also somehow help us to sift hypotheses or generate fresh ideas about the behavior of two-dimensional flows. The problems of instrumentation for such studies are quite acute, and it is even more difficult to attain high Reynolds numbers and desirably thick boundary layers than in studies of two-dimensional flows.

XV. SUGGESTIONS FOR FUTURE WORK

These suggestions do not reflect a review in depth of all important and outstanding problems of the field, but represent a collection of items which appeared to be most interesting and perhaps significant within the scope of this study.

THEORETICAL TOPICS

1. Davey's "strong saddle points of attachment," and their place in a complete flow field. These solutions exhibit a kind of "reverse flow without separation," or "harmless separation." When situated on a leading edge between two adjacent nodes of attachment, the solution for the singular region implies a fascinating flow structure in which the saddle point of the inviscid flow field is transformed into something like an unusually simply-described reattachment node in the surface flow. Whether this really happens in nature is apparently an open question, and a convincing theoretical clarification would probably be heralded as a tour de force in boundary-layer theory.

2. Further development of the three-dimensionalization of Smith and Clutter's numerical procedure. What has been given above is only a very preliminary sketch. Some careful analysis might be worthwhile of finite-difference procedures in the surfaces of constant z, of numerical stability and error control, and some computations for comparison with the method of Raetz.

3. The method of characteristics for solving momentum-integral equations. Mager has brushed off as impractical Timman's original suggestion that the momentum-integral equations be integrated along their real characteristics because the two characteristic directions include too small an angle in the case of weak cross flows. However true this may be, it does not seem proper to close the subject unless one wishes from the start to abandon momentum-integral methods in problems with large and interesting cross flows.

Even for the weak cross-flow case it seems to me that the method of characteristics might be employed in the formulation of an approximate integration method which might be more correct and efficient than

that currently employed. Extending the analysis to cover the method employing both unweighted and weighted integral equations would be a challenging investigation.

EXPERIMENTAL TOPICS

1. Mapping of skin friction field in laminar flow. Serious efforts have been made, particularly in France and England, to observe skin friction trajectories by oil and lampblack, or by dye emission techniques. The results have been very helpful and, in some cases, almost definitive. It would be a major achievement if these or other techniques could be further perfected and employed to generate a definitive portfolio of experimental skin-friction fields, including in particular some which could provide quantitative checks on theory, and some for pedagogical illustration of the richness of qualitative possibilities.

2. Yawing of turbulent wakes. A further understanding of the response of the wake component of turbulent shear flows to gradual and to sudden yawing might contribute significantly to the construction of better models of the three-dimensional turbulent boundary layer. A flat-plate wake might be studied as the wind tunnel downstream of the plate executes a bend around an axis normal to the "plane" of the plate. This would provide an opportunity for the refinement of direction-sensing instrumentation in an environment that is less cramped than the boundary layer.

3. Synoptic exploration of three-dimensional turbulent boundary layers. Careful synoptic measurements of mean-velocity profiles, pressure distributions and surface stresses need to be made on turbulent boundary layers with widely varied inviscid streamline patterns. This almost virgin territory for the experimentalist requires a major commitment to refined experimental technique and to large-scale facilities (to attain the necessary Reynolds number range). Special attention might be paid to cases involving inflected inviscid streamlines and crossover profiles of secondary velocity, which have not as yet been observed.

Appendix

NUMBERED EQUATIONS APPEARING IN TEXT

$$f''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) ff'' + \frac{m}{h_1}(1 - f'^2) + \frac{Vx}{Uy}\frac{1}{h_2}\Big\{\left(s - \frac{n}{2} + \lambda\right) gf''$$

$$+ (n + \lambda)(1 - g'f')\Big\} - \left(\frac{V}{U}\right)^2 \frac{\kappa}{h_1}(1 - g'^2) - 2\left(\frac{\omega_3 LV}{U^2}\right) x(1 - g')$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) + \frac{V}{U}\left(\frac{1}{h_2}\right)\left(g'\frac{\partial f'}{\partial y} + f''\frac{\partial g}{\partial y}\right)\right\} \tag{1}$$

$$g''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) fg'' + \frac{1}{h_1}(r + \kappa)(1 - f'g') + \frac{Vx}{Uy}\frac{1}{h_2}\Big\{\left(s - \frac{n}{2} + \lambda\right) gg''$$

$$+ s(1 - g'^2)\Big\} - \frac{Ux}{Vy}\frac{\lambda}{h_2}(1 - f'^2) + 2\left(\frac{\omega_3 L}{V}\right) x(1 - f')$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial g'}{\partial x} - g''\frac{\partial f}{\partial x}\right) + \frac{V}{U}\left(\frac{1}{h_2}\right)\left(g'\frac{\partial g'}{\partial y} - g''\frac{\partial g}{\partial y}\right)\right\} \tag{2}$$

$$f''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) ff'' + \frac{m}{h_1}(1 - f'^2) + \frac{x}{y}\left(\frac{1}{h_2}\right)\Big\{\left(\frac{n}{2} + \lambda\right) gf''$$

$$- (n + \lambda)G'f'\Big\} + \frac{\kappa}{h_1}G'^2 + 2\left(\frac{\omega_3 L}{U}\right) xG'$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) + \frac{1}{h_2}\left(G'\frac{\partial f'}{\partial y} - f''\frac{\partial G}{\partial x}\right)\right\} \tag{3}$$

$$G''' + \frac{1}{h_1}\left(\frac{m+1}{2} + \kappa\right) fG'' - \left(\frac{m+\kappa}{h_1}\right) f'G' + \frac{x}{y}\left(\frac{1}{h_2}\right)\left\{\left(\frac{n}{2} + \lambda\right) GG''\right.$$

$$\left. - nG'^2 - \lambda(1 - f'^2)\right\} + 2\left(\frac{\omega_3 L}{U}\right) x(1 - f')$$

$$= x\left\{\frac{1}{h_1}\left(f'\frac{\partial G'}{\partial x} - G''\frac{\partial f}{\partial x}\right) + \frac{1}{h_2}\left(G'\frac{\partial G'}{\partial y} - G''\frac{\partial G}{\partial y}\right)\right\} \qquad (4)$$

$$\frac{1}{h_1}\frac{\partial \theta_{11}}{\partial x} + \frac{1}{h_2}\frac{\partial \theta_{12}}{\partial y} = \frac{f''(0)}{x} - \frac{1}{h_1 x}\left\{\left(\frac{3m+1}{2} + 2\kappa\right)\theta_{11} - \kappa(\theta_{11} + \theta_{22}) + m\delta_1\right\}$$

$$- \frac{1}{h_2 y}\left\{\left(\frac{3n}{2} + 2\lambda\right)\theta_{12}\right\} + (\Omega_3 + 2\omega_3)\frac{L}{U}\delta_2 \qquad (5)$$

$$\frac{1}{h_1}\frac{\partial \theta_{21}}{\partial x} + \frac{1}{h_2}\frac{\partial \theta_{22}}{\partial y} = \frac{G''(0)}{x} - \frac{1}{h_1 x}\left\{\left(\frac{3m+1}{2} + 2\kappa\right)\theta_{21}\right\} - (\Omega_3 + 2\omega_3)\frac{L}{U}\delta_1$$

$$- \frac{1}{h_2 y}\left\{\left(\frac{3n}{2} + 2\lambda\right)\theta_{22} - \lambda(\theta_{11} + \theta_{22}) + n\delta_2\right\} \qquad (6)$$

$$A\frac{\partial \delta}{\partial x} + B\frac{\partial \delta}{\partial y} + C\frac{\partial \Delta}{\partial x} + D\frac{\partial \Delta}{\partial y} + E = 0 \qquad (7)$$

$$A'\frac{\partial \delta}{\partial x} + B'\frac{\partial \delta}{\partial y} + C'\frac{\partial \Delta}{\partial x} + D'\frac{\partial \Delta}{\partial y} + E' = 0 \qquad (8)$$

$$a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + C_i = 0, \qquad \left\{\begin{matrix} i = 1,\ 2,\ 3,\ 4 \\ j = 1,\ 2,\ 3,\ 4 \end{matrix}\right\} \qquad (9)$$

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{u}}{\partial \xi} - \frac{U}{h_1}\frac{\partial U}{\partial \xi} + \frac{\bar{v}}{h_2}\frac{\partial \bar{u}}{\partial \zeta} + \bar{w}\frac{\partial \bar{u}}{\partial \zeta} - \bar{v}^2\kappa_2 + \bar{u}\bar{v}\kappa_1 - 2\omega_3\bar{v}$$

$$= \nu\frac{\partial^2 \bar{u}}{\partial \zeta^2} - \frac{\partial}{\partial \zeta}(\overline{u'w'}) - (\overline{u'^2} - \overline{v'^2})\kappa_2 - 2\overline{u'v'}\kappa_1 \qquad (10)$$

$$\frac{\bar{u}}{h_1}\frac{\partial \bar{v}}{\partial \xi} + \frac{\bar{v}}{h_2}\frac{\partial \bar{v}}{\partial \eta} + \bar{w}\frac{\partial \bar{u}}{\partial \zeta} + (U^2 - \bar{u}^2)\varkappa_1 + \bar{u}\bar{v}\varkappa_2 + 2\omega_3\bar{u}$$

$$= \nu \frac{\partial^2 \bar{v}}{\partial \zeta^2} - \frac{\partial}{\partial \zeta}(\overline{v'w'}) - (\overline{v'^2} - \overline{u'^2})\varkappa_1 - 2\overline{u'v'}\,\varkappa_2 \qquad (11)$$

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10. ABSTRACT

The basic concept of a three-dimensional boundary layer is reviewed from both physical and mathematical viewpoints. Emphasis is placed on the various causes of secondary flow, particularly geodesic curvature of the surface streamlines of inviscid flow. Various exact solutions for steady, incompressible laminar flow are reexamined, a proposal for a finite-difference scheme for arbitrary inviscid flows and surface conditions is sketched, and the momentum-integral method and other approximation schemes are briefly discussed. Also considered are compressibility effects, laminar-flow stability, transition to turbulence, displacement thickness of a three-dimensional boundary layer, and the incompressible turbulent boundary layer. It is concluded that very successful three-parameter models of mean velocity profiles exist, but methods for predicting the variation of the profile parameters are essentially deficient. Suggestions for future work include further study of the method of characteristics for solving momentum-integral equations, mapping of the skin friction field in laminar flow, and synoptic exploration of three-dimensional turbulent boundary layers.

11 KEY WORDS

Fluid mechanics
Aerodynamics